

## Section A: Pure Mathematics

- 1** The points  $S$ ,  $T$ ,  $U$  and  $V$  have coordinates  $(s, ms)$ ,  $(t, mt)$ ,  $(u, nu)$  and  $(v, nv)$ , respectively. The lines  $SV$  and  $UT$  meet the line  $y = 0$  at the points with coordinates  $(p, 0)$  and  $(q, 0)$ , respectively. Show that

$$p = \frac{(m-n)sv}{ms-nv},$$

and write down a similar expression for  $q$ .

Given that  $S$  and  $T$  lie on the circle  $x^2 + (y - c)^2 = r^2$ , find a quadratic equation satisfied by  $s$  and by  $t$ , and hence determine  $st$  and  $s + t$  in terms of  $m$ ,  $c$  and  $r$ .

Given that  $S$ ,  $T$ ,  $U$  and  $V$  lie on the above circle, show that  $p + q = 0$ .

- 2** (i) Let  $y = \sum_{n=0}^{\infty} a_n x^n$ , where the coefficients  $a_n$  are independent of  $x$  and are such that this series and all others in this question converge. Show that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and write down a similar expression for  $y''$ .

Write out explicitly each of the three series as far as the term containing  $a_3$ .

- (ii) It is given that  $y$  satisfies the differential equation

$$xy'' - y' + 4x^3y = 0.$$

By substituting the series of part (i) into the differential equation and comparing coefficients, show that  $a_1 = 0$ .

Show that, for  $n \geq 4$ ,

$$a_n = -\frac{4}{n(n-2)} a_{n-4},$$

and that, if  $a_0 = 1$  and  $a_2 = 0$ , then  $y = \cos(x^2)$ .

Find the corresponding result when  $a_0 = 0$  and  $a_2 = 1$ .

**3** The function  $f(t)$  is defined, for  $t \neq 0$ , by

$$f(t) = \frac{t}{e^t - 1}.$$

- (i) By expanding  $e^t$ , show that  $\lim_{t \rightarrow 0} f(t) = 1$ . Find  $f'(t)$  and evaluate  $\lim_{t \rightarrow 0} f'(t)$ .
- (ii) Show that  $f(t) + \frac{1}{2}t$  is an even function. [**Note:** A function  $g(t)$  is said to be *even* if  $g(t) \equiv g(-t)$ .]
- (iii) Show with the aid of a sketch that  $e^t(1-t) \leq 1$  and deduce that  $f'(t) \neq 0$  for  $t \neq 0$ .

Sketch the graph of  $f(t)$ .

**4** For any given (suitable) function  $f$ , the *Laplace transform* of  $f$  is the function  $F$  defined by

$$F(s) = \int_0^{\infty} e^{-st}f(t)dt \quad (s > 0).$$

- (i) Show that the Laplace transform of  $e^{-bt}f(t)$ , where  $b > 0$ , is  $F(s+b)$ .
- (ii) Show that the Laplace transform of  $f(at)$ , where  $a > 0$ , is  $a^{-1}F(\frac{s}{a})$ .
- (iii) Show that the Laplace transform of  $f'(t)$  is  $sF(s) - f(0)$ .
- (iv) In the case  $f(t) = \sin t$ , show that  $F(s) = \frac{1}{s^2 + 1}$ .

Using only these four results, find the Laplace transform of  $e^{-pt} \cos qt$ , where  $p > 0$  and  $q > 0$ .

**5** The numbers  $x$ ,  $y$  and  $z$  satisfy

$$\begin{aligned}x + y + z &= 1 \\x^2 + y^2 + z^2 &= 2 \\x^3 + y^3 + z^3 &= 3.\end{aligned}$$

Show that

$$yz + zx + xy = -\frac{1}{2}.$$

Show also that  $x^2y + x^2z + y^2z + y^2x + z^2x + z^2y = -1$ , and hence that

$$xyz = \frac{1}{6}.$$

Let  $S_n = x^n + y^n + z^n$ . Use the above results to find numbers  $a$ ,  $b$  and  $c$  such that the relation

$$S_{n+1} = aS_n + bS_{n-1} + cS_{n-2},$$

holds for all  $n$ .

**6** Show that  $|e^{i\beta} - e^{i\alpha}| = 2 \sin \frac{1}{2}(\beta - \alpha)$  for  $0 < \alpha < \beta < 2\pi$ . Hence show that

$$|e^{i\alpha} - e^{i\beta}| |e^{i\gamma} - e^{i\delta}| + |e^{i\beta} - e^{i\gamma}| |e^{i\alpha} - e^{i\delta}| = |e^{i\alpha} - e^{i\gamma}| |e^{i\beta} - e^{i\delta}|,$$

where  $0 < \alpha < \beta < \gamma < \delta < 2\pi$ .

Interpret this result as a theorem about cyclic quadrilaterals.

- 7 (i) The functions  $f_n(x)$  are defined for  $n = 0, 1, 2, \dots$ , by

$$f_0(x) = \frac{1}{1+x^2} \quad \text{and} \quad f_{n+1}(x) = \frac{df_n(x)}{dx}.$$

Prove, for  $n \geq 1$ , that

$$(1+x^2)f_{n+1}(x) + 2(n+1)xf_n(x) + n(n+1)f_{n-1}(x) = 0.$$

- (ii) The functions  $P_n(x)$  are defined for  $n = 0, 1, 2, \dots$ , by

$$P_n(x) = (1+x^2)^{n+1}f_n(x).$$

Find expressions for  $P_0(x)$ ,  $P_1(x)$  and  $P_2(x)$ .

Prove, for  $n \geq 0$ , that

$$P_{n+1}(x) - (1+x^2)\frac{dP_n(x)}{dx} + 2(n+1)xP_n(x) = 0,$$

and that  $P_n(x)$  is a polynomial of degree  $n$ .

- 8 Let  $m$  be a positive integer and let  $n$  be a non-negative integer.

- (i) Use the result  $\lim_{t \rightarrow \infty} e^{-mt}t^n = 0$  to show that

$$\lim_{x \rightarrow 0} x^m(\ln x)^n = 0.$$

By writing  $x^x$  as  $e^{x \ln x}$  show that

$$\lim_{x \rightarrow 0} x^x = 1.$$

- (ii) Let  $I_n = \int_0^1 x^m(\ln x)^n dx$ . Show that

$$I_{n+1} = -\frac{n+1}{m+1}I_n$$

and hence evaluate  $I_n$ .

- (iii) Show that

$$\int_0^1 x^x dx = 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{4}\right)^4 + \dots$$

## Section B: Mechanics

**9** A particle is projected under gravity from a point  $P$  and passes through a point  $Q$ . The angles of the trajectory with the positive horizontal direction at  $P$  and at  $Q$  are  $\theta$  and  $\phi$ , respectively. The angle of elevation of  $Q$  from  $P$  is  $\alpha$ .

(i) Show that  $\tan \theta + \tan \phi = 2 \tan \alpha$ .

(ii) It is given that there is a second trajectory from  $P$  to  $Q$  with the same speed of projection. The angles of this trajectory with the positive horizontal direction at  $P$  and at  $Q$  are  $\theta'$  and  $\phi'$ , respectively. By considering a quadratic equation satisfied by  $\tan \theta$ , show that  $\tan(\theta + \theta') = -\cot \alpha$ . Show also that  $\theta + \theta' = \pi + \phi + \phi'$ .

**10** A light spring is fixed at its lower end and its axis is vertical. When a certain particle  $P$  rests on the top of the spring, the compression is  $d$ . When, instead,  $P$  is dropped onto the top of the spring from a height  $h$  above it, the compression at time  $t$  after  $P$  hits the top of the spring is  $x$ . Obtain a second-order differential equation relating  $x$  and  $t$  for  $0 \leq t \leq T$ , where  $T$  is the time at which  $P$  first loses contact with the spring.

Find the solution of this equation in the form

$$x = A + B \cos(\omega t) + C \sin(\omega t),$$

where the constants  $A$ ,  $B$ ,  $C$  and  $\omega$  are to be given in terms of  $d$ ,  $g$  and  $h$  as appropriate.

Show that

$$T = \sqrt{d/g} \left( 2\pi - 2 \arctan \sqrt{2h/d} \right).$$

**11** A comet in deep space picks up mass as it travels through a large stationary dust cloud. It is subject to a gravitational force of magnitude  $Mf$  acting in the direction of its motion. When it entered the cloud, the comet had mass  $M$  and speed  $V$ . After a time  $t$ , it has travelled a distance  $x$  through the cloud, its mass is  $M(1 + bx)$ , where  $b$  is a positive constant, and its speed is  $v$ .

(i) In the case when  $f = 0$ , write down an equation relating  $V$ ,  $x$ ,  $v$  and  $b$ . Hence find an expression for  $x$  in terms of  $b$ ,  $V$  and  $t$ .

(ii) In the case when  $f$  is a non-zero constant, use Newton's second law in the form

$$\text{force} = \text{rate of change of momentum}$$

to show that

$$v = \frac{ft + V}{1 + bx}.$$

Hence find an expression for  $x$  in terms of  $b$ ,  $V$ ,  $f$  and  $t$ .

Show that it is possible, if  $b$ ,  $V$  and  $f$  are suitably chosen, for the comet to move with constant speed. Show also that, if the comet does not move with constant speed, its speed tends to a constant as  $t \rightarrow \infty$ .

## Section C: Probability and Statistics

- 12 (i)** Albert tosses a fair coin  $k$  times, where  $k$  is a given positive integer. The number of heads he gets is  $X_1$ . He then tosses the coin  $X_1$  times, getting  $X_2$  heads. He then tosses the coin  $X_2$  times, getting  $X_3$  heads. The random variables  $X_4, X_5, \dots$  are defined similarly. Write down  $E(X_1)$ .

By considering  $E(X_2 \mid X_1 = x_1)$ , or otherwise, show that  $E(X_2) = \frac{1}{4}k$ .

Find  $\sum_{i=1}^{\infty} E(X_i)$ .

- (ii)** Bertha has  $k$  fair coins. She tosses the first coin until she gets a tail. The number of heads she gets before the first tail is  $Y_1$ . She then tosses the second coin until she gets a tail and the number of heads she gets with this coin before the first tail is  $Y_2$ . The random variables  $Y_3, Y_4, \dots, Y_k$  are defined similarly, and  $Y = \sum_{i=1}^k Y_i$ .

Obtain the probability generating function of  $Y$ , and use it to find  $E(Y)$ ,  $\text{Var}(Y)$  and  $P(Y = r)$ .

- 13 (i)** The point  $P$  lies on the circumference of a circle of unit radius and centre  $O$ . The angle,  $\theta$ , between  $OP$  and the positive  $x$ -axis is a random variable, uniformly distributed on the interval  $0 \leq \theta < 2\pi$ . The cartesian coordinates of  $P$  with respect to  $O$  are  $(X, Y)$ . Find the probability density function for  $X$ , and calculate  $\text{Var}(X)$ .

Show that  $X$  and  $Y$  are uncorrelated and discuss briefly whether they are independent.

- (ii)** The points  $P_i$  ( $i = 1, 2, \dots, n$ ) are chosen independently on the circumference of the circle, as in part (i), and have cartesian coordinates  $(X_i, Y_i)$ . The point  $\bar{P}$  has coordinates  $(\bar{X}, \bar{Y})$ , where  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$  and  $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$ . Show that  $\bar{X}$  and  $\bar{Y}$  are uncorrelated.

Show that, for large  $n$ ,  $P\left(|\bar{X}| \leq \sqrt{\frac{2}{n}}\right) \approx 0.95$ .