1 **(i)** By considering the series expansion of $(x^2 + 5x + 4)\,e^x$, or otherwise, show that

$$10\,e = 4 + \frac{3^2}{1!} + \frac{4^2}{2!} + \frac{5^2}{3!} + \cdots .$$

(ii) Show that

$$5\,e = 1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots .$$

(iii) Evaluate

$$1 + \frac{2^3}{1!} + \frac{3^3}{2!} + \frac{4^3}{3!} + \cdots .$$

2 Let

$$f(t) = \frac{\ln t}{t} \quad \text{for } t > 0.$$

Sketch the graph of $f(t)$ and find its maximum value. How many values of t correspond to a given positive value of $f(t)$?

Find how many positive values of y satisfy $x^y = y^x$ for a given positive value of x. Sketch the set of points (x, y) which satisfy $x^y = y^x$ with $x, y > 0$.

3 By considering the solutions of the equation $z^n - 1 = 0$, or otherwise, show that

$$(z - \omega)(z - \omega^2)\ldots(z - \omega^{n-1}) = 1 + z + z^2 + \cdots + z^{n-1},$$

where z is any complex number and $\omega = e^{2\pi i/n}$.

Let $A_1, A_2, A_3, \ldots, A_n$ be points equally spaced around a circle of radius r centred at O (so that they are the vertices of a regular n-sided polygon).

Show that

$$\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \cdots + \overrightarrow{OA_n} = \mathbf{0}.$$

Deduce, or prove otherwise, that

$$\sum_{k=1}^{n} |A_1 A_k|^2 = 2r^2 n.$$

4 In this question, you may assume that if $k_1, \ldots, k_n$ are distinct positive real numbers, then

$$\frac{1}{n}\sum_{r=1}^{n} k_r > \left(\prod_{r=1}^{n} k_r\right)^{\frac{1}{n}},$$

i.e. their arithmetic mean is greater than their geometric mean.

Suppose that a, b, c and d are positive real numbers such that the polynomial

$$\mathrm{f}(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$$

has four distinct positive roots p, q, r and s.

(i) Show that pqr, qrs, rsp, and spq are distinct.

(ii) By considering the relationship between the coefficients of f and its roots, show that $c > d$.

(iii) Explain why the polynomial $\mathrm{f}'(x)$ must have three distinct roots.

(iv) By differentiating f, show that $b > c$.

(v) Show that $a > b$.

5 Find the ratio, over one revolution, of the distance moved by a wheel rolling on a flat surface to the distance traced out by a point on its circumference.

6 Suppose that y_n satisfies the equations

$$(1-x^2)\frac{\mathrm{d}^2y_n}{\mathrm{d}x^2} - x\frac{\mathrm{d}y_n}{\mathrm{d}x} + n^2y_n = 0,$$

$$y_n(1) = 1, \quad y_n(x) = (-1)^n y_n(-x).$$

If $x = \cos\theta$, show that

$$\frac{\mathrm{d}^2y_n}{\mathrm{d}\theta^2} + n^2y_n = 0,$$

and hence obtain y_n as a function of θ. Deduce that for $|x| \leqslant 1$

$$y_0 = 1, \quad y_1 = x,$$

$$y_{n+1} - 2xy_n + y_{n-1} = 0.$$

7 For each positive integer n, let

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots ;$$
$$b_n = \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots .$$

(i) Evaluate b_n.

(ii) Show that $0 < a_n < 1/n$.

(iii) Deduce that $a_n = n!\mathrm{e} - [n!\mathrm{e}]$ (where $[x]$ is the integer part of x).

(iv) Hence show that e is irrational.

8 Let R_α be the 2×2 matrix that represents a rotation through the angle α and let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

(i) Find in terms of a, b and c an angle α such that the matrix $R_{-\alpha}AR_\alpha$ is diagonal (i.e. has the value zero in top-right and bottom-left positions).

(ii) Find values of a, b and c such that the equation of the ellipse

$$x^2 + (y + 2x\cot 2\theta)^2 = 1 \quad (0 < \theta < \frac{\pi}{4})$$

can be expressed in the form

$$(x \quad y)A\begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

Show that, for this A, the matrix $R_{-\alpha}AR_\alpha$ is diagonal if $\alpha = \theta$. Express the non–zero elements of this diagonal matrix in terms of θ.

(iii) Deduce, or show otherwise, that the minimum and maximum distances from the centre to the circumference of this ellipse are $\tan\theta$ and $\cot\theta$.

9 A uniform rigid rod BC is suspended from a fixed point A by light stretched springs AB, AC. The springs are of different natural lengths but the ratio of tension to extension is the same constant κ for each. The rod is *not* hanging vertically. Show that the ratio of the lengths of the stretched springs is equal to the ratio of the natural lengths of the unstretched springs.

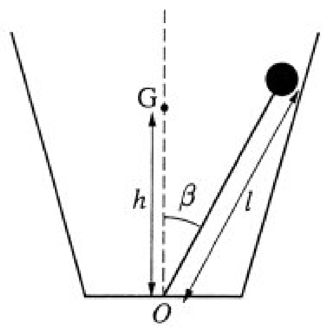
10 By pressing a finger down on it, a uniform spherical marble of radius a is made to slide along a horizontal table top with an initial linear speed v_0 and an initial *backward* angular speed ω_0 about the horizontal axis perpendicular to v_0. The frictional force between the marble and the table is constant (independent of speed).

Find the value of $v_0/(a\omega_0)$ for which the marble

(i) slides to a complete stop,

(ii) comes to a stop and then rolls back towards its initial position with linear speed $v_0/7$.

11



A heavy symmetrical bell and clapper can both swing freely in a vertical plane about a point O on a horizontal beam at the apex of the bell. The mass of the bell is M and its moment of inertia about the beam is Mk^2. Its centre of mass, G, is a distance h from O. The clapper may be regarded as a small heavy ball on a light rod of length l. Initially the bell is held with its axis vertical and its mouth above the beam. The clapper ball rests against the side of the bell, with the rod making an angle β with the axis. The bell is then released. Show that, at the moment when the clapper and bell separate, the clapper rod makes an angle α with the upward vertical, where

$$\cot\alpha = \cot\beta - \frac{k^2}{hl}\operatorname{cosec}\beta\,.$$

12 **(i)** I toss a biased coin which has a probability p of landing heads and a probability $q = 1 - p$ of landing tails. Let K be the number of tosses required to obtain the first head and let

$$G(s) = \sum_{k=1}^{\infty} P(K = k)s^k.$$

Show that

$$G(s) = \frac{ps}{1 - qs}$$

and hence find the expectation and variance of K.

(ii) I sample cards at random with replacement from a normal pack of 52. Let N be the total number of draws I make in order to sample every card at least once. By expressing N as a sum $N = N_1 + N_2 + \cdots + N_{52}$ of random variables, or otherwise, find the expectation of N. Estimate the numerical value of this expectation, using the approximations $e \approx 2.7$ and $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx 0.5 + \ln n$ if n is large.

13 Let X and Y be independent standard normal random variables. The probability density function, f, of each is therefore given by

$$f(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}.$$

(i) Find the moment generating function $E(e^{\theta X})$ of X.

(ii) Find the moment generating function of $aX + bY$ and hence obtain the condition on a and b which ensures that $aX + bY$ has the same distribution as X and Y.

(iii) Let $Z = e^{\mu + \sigma X}$. Show that

$$E(Z^\theta) = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2},$$

and hence find the expectation and variance of Z.

14 An industrial process produces rectangular plates of mean length μ_1 and mean breadth μ_2. The length and breadth vary independently with non-zero standard deviations σ_1 and σ_2 respectively. Find the means and standard deviations of the perimeter and of the area of the plates. Show that the perimeter and area are not independent.