1 A cylindrical biscuit tin has volume V and surface area S (including the ends). Show that the minimum possible surface area for a given value of V is $S = 3(2\pi V^2)^{1/3}$. For this value of S show that the volume of the largest sphere which can fit inside the tin is $\frac{2}{3}V$, and find the volume of the smallest sphere into which the tin fits.

2 (i) Show that

$$\int_0^1 (1 + (\alpha - 1)x)^n \mathrm{d}x = \frac{\alpha^{n+1} - 1}{(n+1)(\alpha - 1)}$$

when $\alpha \neq 1$ and n is a positive integer.

(ii) Show that if $0 \leq k \leq n$ then the coefficient of α^k in the polynomial

$$\int_0^1 (\alpha x + (1 - x))^n \mathrm{d}x$$

is

$$\binom{n}{k} \int_0^1 x^k (1 - x)^{n-k} \mathrm{d}x.$$

(iii) Hence, or otherwise, show that

$$\int_0^1 x^k (1 - x)^{n-k} \mathrm{d}x = \frac{k!(n-k)!}{(n+1)!}.$$

3 Let n be a positive integer.

(i) Factorize $n^5 - n^3$, and show that it is divisible by 24.

(ii) Prove that $2^{2n} - 1$ is divisible by 3.

(iii) If $n - 1$ is divisible by 3, show that $n^3 - 1$ is divisible by 9.

4 Show that

$$\int_0^1 \frac{1}{x^2 + 2ax + 1} \, \mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{1 - a^2}} \tan^{-1} \sqrt{\dfrac{1 - a}{1 + a}} & \text{if } |a| < 1, \\ \dfrac{1}{2\sqrt{a^2 - 1}} \ln |a + \sqrt{a^2 - 1}| & \text{if } |a| > 1. \end{cases}$$

5 (i) Find all rational numbers r and s which satisfy

$$(r + s\sqrt{3})^2 = 4 - 2\sqrt{3}.$$

(ii) Find all real numbers p and q which satisfy

$$(p + qi)^2 = (3 - 2\sqrt{3}) + 2(1 - \sqrt{3})i.$$

(iii) Solve the equation

$$(1 + i)z^2 - 2z + 2\sqrt{3} - 2 = 0,$$

writing your solutions in as simple a form as possible.

[No credit will be given to answers involving use of calculators.]

6 Let $\mathrm{f}(x) = \dfrac{\sin(n + \frac{1}{2})x}{\sin(x/2)}$ for $0 < x \leqslant \pi$.

(i) Using the formula

$$2\sin(x/2)\cos kx = \sin(k + \tfrac{1}{2})x - \sin(k - \tfrac{1}{2})x$$

(which you may assume), or otherwise, show that

$$\mathrm{f}(x) = 1 + 2\sum_{k=1}^{n} \cos kx.$$

(ii) Find $\displaystyle\int_0^\pi \mathrm{f}(x)\,\mathrm{d}x$ and $\displaystyle\int_0^\pi \mathrm{f}(x)\cos x\,\mathrm{d}x$.

7 (i) At time $t = 0$ a tank contains one unit of water. Water flows out of the tank at a rate proportional to the amount of water in the tank. The amount of water in the tank at time t is y. Show that there is a constant $b < 1$ such that $y = b^t$.

(ii) Suppose instead that the tank contains one unit of water at time $t = 0$, but that in addition to water flowing out as described, water is added at a steady rate $a > 0$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}t} - y\ln b = a,$$

and hence find y in terms of a, b and t.

8 (i) By using the formula for the sum of a geometric series, or otherwise, express the number 0.38383838... as a fraction in its lowest terms.

(ii) Let x be a real number which has a recurring decimal expansion

$$x = 0 \cdot a_1 a_2 a_3 \dots ,$$

so that there exist positive integers N and k such that $a_{n+k} = a_n$ for all $n > N$. Show that

$$x = \frac{b}{10^N} + \frac{c}{10^N(10^k - 1)},$$

where b and c are integers to be found. Deduce that x is rational.

Section B: Mechanics

9 A bungee-jumper of mass m is attached by means of a light rope of natural length l and modulus of elasticity $\frac{mg}{k}$, where k is a constant, to a bridge over a ravine. She jumps from the bridge and falls vertically towards the ground. Ignoring air resistance, find her speed when the rope becomes taut. If she only just avoids hitting the ground, show that the height h of the bridge above the floor of the ravine satisfies

$$h^2 - 2hl(k+1) + l^2 = 0,$$

and hence find h. Show that the maximum speed v which she attains during her fall satisfies

$$v^2 = (k+2)gl.$$

10 A spaceship of mass M is at rest. It separates into two parts in an explosion in which the total kinetic energy released is E. Immediately after the explosion the two parts have masses m_1 and m_2 and speeds v_1 and v_2 respectively. Show that the minimum possible relative speed $v_1 + v_2$ of the two parts of the spaceship after the explosion is $(8E/M)^{1/2}$.

11 A particle is projected under the influence of gravity from a point O on a level plane in such a way that, when its horizontal distance from O is c, its height is h. It then lands on the plane at a distance $c + d$ from O. Show that the angle of projection α satisfies

$$\tan \alpha = \frac{h(c + d)}{cd}$$

and that the speed of projection v satisfies

$$v^2 = \frac{g}{2}\left(\frac{cd}{h} + \frac{(c + d)^2 h}{cd}\right).$$

Section C: Probability and Statistics

12 An examiner has to assign a mark between 1 and m inclusive to each of n examination scripts ($n \leqslant m$). He does this randomly, but never assigns the same mark twice. If K is the highest mark that he assigns, explain why

$$\mathrm{P}(K = k) = \binom{k-1}{n-1} \Big/ \binom{m}{n}$$

for $n \leqslant k \leqslant m$, and deduce that

$$\sum_{k=n}^{m} \binom{k-1}{n-1} = \binom{m}{n}.$$

Find the expected value of K.

13 I have a Penny Black stamp which I want to sell to my friend Jim, but we cannot agree a price. So I put the stamp under one of two cups, jumble them up, and let Jim guess which one it is under. If he guesses correctly, I add a third cup, jumble them up, and let Jim guess again. I repeat the process until Jim fails to guess correctly, adding another cup each time. The price he pays for the stamp is £N, where N is the number of cups present when Jim fails to guess correctly. Find $\mathrm{P}(N = k)$. Show that $\mathrm{E}(N) = \mathrm{e}$ and calculate $\mathrm{Var}(N)$.

14 A biased coin, with a probability p of coming up heads and a probability $q = 1 - p$ of coming up tails, is tossed repeatedly. Let A be the event that the first run of r successive heads occurs before the first run of s successive tails. If H is the event that on the first toss the coin comes up heads and T is the event that it comes up tails, show that

$$\mathrm{P}(A|H) = p^{\alpha} + (1 - p^{\alpha})\mathrm{P}(A|T),$$
$$\mathrm{P}(A|T) = (1 - q^{\beta})\mathrm{P}(A|H),$$

where α and β are to be determined. Use these two equations to find $\mathrm{P}(A|H)$, $\mathrm{P}(A|T)$, and hence $\mathrm{P}(A)$.