

- 1 (i) Find the real values of x for which

$$x^3 - 4x^2 - x + 4 \geq 0.$$

- (ii) Find the three lines in the (x, y) plane on which

$$x^3 - 4x^2y - xy^2 + 4y^3 = 0.$$

- (iii) On a sketch shade the regions of the (x, y) plane for which

$$x^3 - 4x^2y - xy^2 + 4y^3 \geq 0.$$

- 2 (i) Suppose that

$$S = \int \frac{\cos x}{\cos x + \sin x} dx \quad \text{and} \quad T = \int \frac{\sin x}{\cos x + \sin x} dx.$$

By considering $S + T$ and $S - T$ determine S and T .

- (ii) Evaluate $\int_{\frac{1}{4}}^{\frac{1}{2}} (1 - 4x)\sqrt{\left(\frac{1}{x} - 1\right)} dx$ by using the substitution $x = \sin^2 t$.

- 3 (i) If $f(r)$ is a function defined for $r = 0, 1, 2, 3, \dots$, show that

$$\sum_{r=1}^n \{f(r) - f(r-1)\} = f(n) - f(0).$$

- (ii) If $f(r) = r^2(r+1)^2$, evaluate $f(r) - f(r-1)$ and hence determine $\sum_{r=1}^n r^3$.

- (iii) Find the sum of the series $1^3 - 2^3 + 3^3 - 4^3 + \dots + (2n+1)^3$.

4 By applying de Moivre's theorem to $\cos 5\theta + i \sin 5\theta$, expanding the result using the binomial theorem, and then equating imaginary parts, show that

$$\sin 5\theta = \sin \theta(16 \cos^4 \theta - 12 \cos^2 \theta + 1).$$

Use this identity to evaluate $\cos^2 \frac{\pi}{5}$, and deduce that $\cos \frac{\pi}{5} = \frac{1}{4}(1 + \sqrt{5})$.

5 If

$$f(x) = nx - \binom{n}{2} \frac{x^2}{2} + \binom{n}{3} \frac{x^3}{3} - \dots + (-1)^{r+1} \binom{n}{r} \frac{x^r}{r} + \dots + (-1)^{n+1} \frac{x^n}{n},$$

show that

$$f'(x) = \frac{1 - (1-x)^n}{x}.$$

Deduce that

$$f(x) = \int_{1-x}^1 \frac{1-y^n}{1-y} dy.$$

Hence show that

$$f(1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

6 (i) In the differential equation

$$\frac{1}{y^2} \frac{dy}{dx} + \frac{1}{y} = e^{2x}$$

make the substitution $u = 1/y$, and hence show that the general solution of the original equation is

$$y = \frac{1}{Ae^x - e^{2x}}.$$

(ii) Use a similar method to solve the equation

$$\frac{1}{y^3} \frac{dy}{dx} + \frac{1}{y^2} = e^{2x}.$$

7 Let A, B, C be three non-collinear points in the plane. Explain briefly why it is possible to choose an origin equidistant from the three points. Let O be such an origin, let G be the centroid of the triangle ABC , let Q be a point such that $\overrightarrow{GQ} = 2\overrightarrow{OG}$, and let N be the midpoint of OQ .

(i) Show that \overrightarrow{AQ} is perpendicular to \overrightarrow{BC} and deduce that the three altitudes of $\triangle ABC$ are concurrent.

(ii) Show that the midpoints of AQ , BQ and CQ , and the midpoints of the sides of $\triangle ABC$ are all equidistant from N .

[The centroid of $\triangle ABC$ is the point G such that $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$. The altitudes of the triangle are the lines through the vertices perpendicular to the opposite sides.]

8 Find functions f , g and h such that the equation

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)y = h(x) \quad (*)$$

is satisfied by all three of the solutions $y = x$, $y = 1$ and $y = x^{-1}$ for $0 < x < 1$.

If f, g and h are the functions you have found in the first paragraph, what condition must the real numbers a , b and c satisfy in order that

$$y = ax + b + \frac{c}{x}$$

should be a solution of (*)?

Section B: Mechanics

9 A particle is projected from a point O with speed $\sqrt{(2gh)}$, where g is the acceleration due to gravity. Show that it is impossible, whatever the angle of projection, for the particle to reach a point above the parabola

$$x^2 = 4h(h - y),$$

where x is the horizontal distance from O and y is the vertical distance above O . State briefly the simplifying assumptions which this solution requires.

10 A small ball of mass m is suspended in equilibrium by a light elastic string of natural length l and modulus of elasticity λ . Show that the total length of the string in equilibrium is $l(1 + \frac{mg}{\lambda})$.

If the ball is now projected downwards from the equilibrium position with speed u_0 , show that the speed v of the ball at distance x below the equilibrium position is given by

$$v^2 + \frac{\lambda}{lm}x^2 = u_0^2.$$

At distance h , where $\lambda h^2 < lmu_0^2$, below the equilibrium position is a horizontal surface on which the ball bounces with a coefficient of restitution e . Show that after one bounce the velocity u_1 at $x = 0$ is given by

$$u_1^2 = e^2u_0^2 + \frac{\lambda}{lm}h^2(1 - e^2),$$

and that after the second bounce the velocity u_2 at $x = 0$ is given by

$$u_2^2 = e^4u_0^2 + \frac{\lambda}{lm}h^2(1 - e^4).$$

11 Two identical uniform cylinders, each of mass m , lie in contact with one another on a horizontal plane and a third identical cylinder rests symmetrically on them in such a way that the axes of the three cylinders are parallel. Assuming that all the surfaces in contact are equally rough, show that the minimum possible coefficient of friction is $2 - \sqrt{3}$.

12 A school has n pupils, of whom r play hockey, where $n \geq r \geq 2$. All n pupils are arranged in a row at random.

- (i) What is the probability that there is a hockey player at each end of the row?
- (ii) What is the probability that all the hockey players are standing together?
- (iii) By considering the gaps between the non-hockey-players, find the probability that no two hockey players are standing together, distinguishing between cases when the probability is zero and when it is non-zero.

13 A scientist is checking a sequence of microscope slides for cancerous cells, marking each cancerous cell that she detects with a red dye. The number of cancerous cells on a slide is random and has a Poisson distribution with mean μ . The probability that the scientist spots any one cancerous cell is p , and is independent of the probability that she spots any other one.

- (i) Show that the number of cancerous cells which she marks on a single slide has a Poisson distribution of mean $p\mu$.
- (ii) Show that the probability Q that the second cancerous cell which she marks is on the k th slide is given by

$$Q = e^{-\mu p(k-1)} \{ (1 + k\mu p)(1 - e^{-\mu p}) - \mu p \}.$$

- 14** (i) Find the maximum value of $\sqrt{p(1-p)}$ as p varies between 0 and 1.
- (ii) Suppose that a proportion p of the population is female. In order to estimate p we pick a sample of n people at random and find the proportion of them who are female. Find the value of n which ensures that the chance of our estimate of p being more than 0.01 in error is less than 1%.
- (iii) Discuss how the required value of n would be affected if (a) p were the proportion of people in the population who are left-handed; (b) p were the proportion of people in the population who are millionaires.