Proof That e Is Irrational

Preliminaries: We require knowledge that

$$e^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!} \equiv 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

and therefore

$$e \equiv 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots .$$

As with many irrationality proofs we suppose that e is rational for contradiction. Therefore suppose

$$e = \frac{p}{q} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

where p and q are integers. Since q is an integer we must somewhere get to the term $\frac{1}{q!}$ in the series for e, so

$$\frac{p}{q} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \cdots .$$

Multiplying both sides by $q!$ we obtain

$$\begin{aligned} q! \times \frac{p}{q} &= q!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \cdots\right) \\ p(q-1)! &= q!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right) + \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \cdots . \end{aligned}$$

The term $p(q-1)!$ is clearly an integer. The term $q!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right)$ is also an integer since $q!$ is divisible by all factorials up to, and including, $q!$. So if we can demonstrate that the remaining term $\frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \cdots$ is *not* an integer then our proof is complete, since it is impossible that integer $=$ integer $+$ non-integer.

Now

$$\frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \frac{q!}{(q+3)!} + \cdots = \frac{1}{q+1} + \frac{1}{(q+2)(q+1)} + \frac{1}{(q+3)(q+2)(q+1)} + \cdots$$

and we can see (by considering respective terms) that

$$\frac{1}{q+1} + \frac{1}{(q+2)(q+1)} + \frac{1}{(q+3)(q+2)(q+1)} + \cdots < \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots .$$

The left hand side of the above is clearly greater than zero. The right hand side

$$\frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots$$

is an infinite geometric series with

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{q+1}}{1 - \frac{1}{q+1}} = \frac{1}{q} < 1.$$

[Note S_∞ exists since $r = \frac{1}{q+1}$ clearly satisfies $-1 < r < 1$.] Therefore

$$0 < \frac{1}{q+1} + \frac{1}{(q+2)(q+1)} + \frac{1}{(q+3)(q+2)(q+1)} + \cdots < 1$$

which demonstrates that $\frac{1}{q+1} + \frac{1}{(q+2)(q+1)} + \frac{1}{(q+3)(q+2)(q+1)} + \cdots$ is not an integer and our contradiction is complete.

$$e \neq \frac{p}{q} \qquad \text{for integer } p \text{ and } q.$$

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