Proof That $\sqrt{2}$ Is Irrational

This document proves that $\sqrt{2}$ is irrational (i.e. one which can’t be expressed as a fraction of one integer over another). The technique used is one of proof by contradiction. It is a technique widely used by mathematicians, but most A Level students will not have seen it.

**Proof**

We are trying to prove that $\sqrt{2}$ cannot be expressed as a fraction. If we are trying to prove that something *cannot* be true, it is often useful to assume that it *is* true and attempt to prove a contradiction. So let us assume that

$$\sqrt{2} = \frac{a}{b}$$

where $\frac{a}{b}$ is a fraction in its lowest form.

Let us play around with this formula and see what we can come up with.

$$\sqrt{2} = \frac{a}{b}$$

$$2 = \frac{a^2}{b^2} \quad \text{squaring both sides}$$

$$2b^2 = a^2 \quad \text{multiplying by } b^2$$

So $a^2$ is an even number $\Rightarrow$ $a$ is an even number. We can therefore express $a$ as $2c$ where $c$ is also an integer.

$$2b^2 = a^2$$

$$2b^2 = (2c)^2 \quad \text{substituting } 2c \text{ for } a$$

$$2b^2 = 4c^2 \quad \text{getting rid of brackets}$$

$$b^2 = 2c^2 \quad \text{cancelling the } 2$$

We can now see that $b^2$ is also an even number $\Rightarrow$ $b$ is even.

But we have assumed that $\frac{a}{b}$ is a fraction in its lowest form, which it clearly is not since both $a$ and $b$ are even numbers (and could therefore be cancelled further). So we have a contradiction and have to conclude that our original assumption that $\sqrt{2}$ can be expressed as a fraction is false $\Rightarrow$ $\sqrt{2}$ is irrational. ■