Proof That $\sqrt{2}$ Is Irrational

This document proves that $\sqrt{2}$ is irrational (i.e. one which can't be expressed as a fraction of one integer over another). The technique used is one of *proof by contradiction*. It is a technique widely used by mathematicians, but most A Level students will not have seen it.

Proof

We are trying to prove that $\sqrt{2}$ cannot be expressed as a fraction. If we are trying to prove that something *cannot* be true, it is often useful to assume that it *is* true and attempt to prove a contradiction. So let us assume that

$$\sqrt{2} = \frac{a}{b}$$

where $\frac{a}{b}$ is a fraction in its lowest form.

Let us play around with this formula and see what we can come up with.

$$\begin{aligned} \sqrt{2} &= \frac{a}{b} \\ 2 &= \frac{a^2}{b^2} && \text{squaring both sides} \\ 2b^2 &= a^2 && \text{multiplying by } b^2 \end{aligned}$$

So a^2 is an even number $\Rightarrow a$ is an even number. We can therefore express a as $2c$ where c is also an integer.

$$\begin{aligned} 2b^2 &= a^2 \\ 2b^2 &= (2c)^2 && \text{substituting } 2c \text{ for } a \\ 2b^2 &= 4c^2 && \text{getting rid of brackets} \\ b^2 &= 2c^2 && \text{cancelling the 2} \end{aligned}$$

We can now see that b^2 is also and even number $\Rightarrow b$ is even.

But we have assumed that $\frac{a}{b}$ is a fraction in its lowest form, which it clearly is not since both a and b are even numbers (and could therefore be cancelled further). So we have a contradiction and have to conclude that our original assumption that $\sqrt{2}$ can be expressed as a fraction is false $\Rightarrow \sqrt{2}$ is irrational. ■