Three questions from unknown Oxford Entrance Paper (prior to 1996)

B6. Let G be the set of 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are real numbers such that $ad - bc = 1$.

Let $t(A) = a + d$.

(i) Let $A, B \in G$. Show that

$$t(AB) + t(AB^{-1}) = t(A)t(B).$$

(ii) Let $A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $x = t(A)$.

By writing $A^n = A^{n-1}A$ for $n \geq 2$ and using part (i) show that $t(A^n) = xt(A^{n-1}) - t(A^{n-2})$ for $n \geq 2$.

(iii) By using induction (or otherwise) prove that

$$t(A^n) = \left(\frac{x + \sqrt{x^2 - 4}}{2}\right)^n + \left(\frac{x - \sqrt{x^2 - 4}}{2}\right)^n.$$

C11. Let $a > 0$ and let $f(x) = \dfrac{x^2 + a^2}{x^2 - ax}$.

(i) Find the turning points of the graph of $f(x)$ and find the values of $f(x)$ at these turning points.

(ii) Find the asymptotes of the graph.

(iii) Sketch the graph, marking the features you have found in (i) and (ii). Your sketch should make clear how the graph approaches the asymptotes.

C12. Let $f(x) = e^{2x + x^2}$

(i) Show that for small values of x, $f(x)$ can be approximated by $1 + 2x + 3x^2 + \frac{10}{3}x^3$, neglecting higher powers of x.

(ii) Let $g(x) = 1 + 2x + 3x^2$, and show that $f(x) - g(x) = \frac{10}{3}x^3(1 + \sum_{n=1}^{\infty} a_n x^n)$ where $a_n \geq 0$ for all n.

(iii) Given that $f(0.1) - g(0.1) < 0.004$, show that if $0 \leq x \leq 0.1$ then $f(x) - g(x) \leq 4x^3$.

(iv) Prove that $\int_0^{0.1} g(x)dx$ differs from $\int_0^{0.1} f(x)dx$ by at most 10^{-4}.