SOLUTIONS FOR ADMISSIONS TEST IN
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
WEDNESDAY 7 NOVEMBER 2012

Mark Scheme:

Each part of Question 1 is worth four marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. One can proceed by elimination. Point $(1,1)$ lies on line (a) and is clearly inside the given circle. $\left(\sqrt{2},0\right)$ lies on both lines (c) and (d) which is again inside the given circle. **The answer is (b).**

B. We can rewrite N as

$$N = 2^k \times 4^m \times 8^n = 2^{k+2m+3n}.$$

Now 2^r is a square when r is even and is not a square when r is odd. So we need

$$k + 2m + 3n = k + n + 2\,(m+n)$$

to be even , which is equivalent to needing $k+n$ to be even. **The answer is (d).**

C. Note that $\log_3 \left(9^2\right) = 2\log_3 9 = 4$. By comparison with this number

$$\begin{aligned} \left(\sqrt{3}\right)^3 &= 3\sqrt{3} > 4 \qquad \text{as on squaring } 27 > 16; \\ \left(3\sin\frac{\pi}{3}\right)^2 &= \left(\frac{3\sqrt{3}}{2}\right)^2 = \frac{27}{4} > 4; \\ \log_2\left(\log_2\left(8^5\right)\right) &= \log_2\left(5\log_2 8\right) = \log_2 15 < \log_2 16 = 4. \end{aligned}$$

The answer is (d).

D. If $-1 < c < 0$ then the horizontal and vertical sides of the shaded triangle have length $1+c$ and hence

$$A(c) = \frac{1}{2}\left(1+c\right)^2.$$

This is a parabolic curve with minimum at $(-1,0)$ and so **the answer is (a).**

E. The curve is that of a quintic with a negative leading coefficient which eliminates (c). Two of its roots are repeated which eliminates (b). Finally the smallest and middle of these roots are the repeated ones which eliminates (a) which has its smallest and largest roots repeated. **The answer is (d).**

F. None of the integrals needs to be calculated. We need only note

$$\int_{-\pi/2}^{\pi/2} \cos x \, dx \quad > \quad 0 \quad \text{as } \cos x > 0 \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2};$$
$$\int_{\pi}^{2\pi} \sin x \, dx \quad < \quad 0 \quad \text{as } \sin x < 0 \text{ for } \pi < x < 2\pi;$$

and

$$\int_0^{\pi/8} \frac{dx}{\cos 3x} \text{ is defined and positive as } \cos 3x > \cos\frac{3\pi}{8} > 0 \text{ for } 0 < 3x < \frac{3\pi}{8};$$

The product of two positive numbers and a negative one is negative. **The answer is (b).**

G. Using the second equation to eliminate x we see that

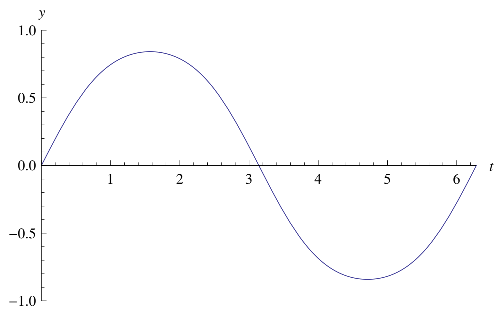
$$2(k-y) + ky = 4 \implies (k-2)\,y = (4-2k) = -2(k-2).$$

If $k \neq 2$ then we can divide by $k-2$ and we see $y = -2$. So positive solutions aren't possible when $k \neq 2$. When $k = 2$ then the equations are

$$2x + 2y = 4, \qquad\qquad x + y = 2$$

which are both clearly satisfied by $x = y = 1$. **The answer is (c).**

H. A sketch of $y = \sin(\sin t)$ in the range $0 < t \leqslant 2\pi$ looks like



To appreciate this we need to realise how the graph $y = \sin(\sin t)$ relates to the graph $y = \sin t$. The value $\sin t$ is in the range $-1 \leqslant \sin t \leqslant 1$ and for values in the range $-1 \leqslant \theta \leqslant 1$ then $\sin\theta$ is a value which is smaller than but of the same sign as θ. Importantly sin is also odd.

So for $0 < x \leqslant \pi$ then

$$\int_0^x \sin(\sin t)\, dt > 0.$$

Also, by the oddness of sin we have that

$$\int_{\pi}^{2\pi} \sin(\sin t)\, dt = -\int_0^{\pi} \sin(\sin t)\, dt.$$

So

$$\int_0^x \sin(\sin t)\, dt$$

remains positive for $\pi < x < 2\pi$, the integral only becoming 0 when the signed area between π and 2π cancels out the area between 0 and π. The only root of the equation in the given range is $x = 2\pi$ and **the answer is (b).**

I. If the radius of the circle is r then we have $2\pi r = 10$ and $r = 5/\pi$. This distance r is also the distance from the centre of the triangle to any of its vertices. So we have

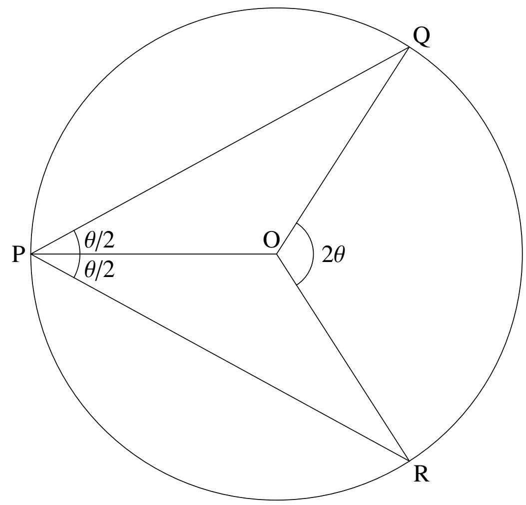
$$A = 3 \times \frac{1}{2} r^2 \sin\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{4} r^2; \qquad P = 6 \times r \sin\left(\frac{\pi}{3}\right) = 3\sqrt{3} r.$$

Hence

$$\frac{A}{P} = \frac{3\sqrt{3}r^2/4}{3\sqrt{3}r} = \frac{r}{4} = \frac{5}{4\pi}$$

and **the answer is (a).**

J. The area QPR is largest when Q and R are symmetrically placed about P, for if (say) PQ were longer than PR then Q could be moved so as to gain more area than would be lost by the corresponding move of R. This means that the angles QPO and RPO are both $\theta/2$; the angle QOR is 2θ as the angle subtended by QR at the centre O is twice that subtended at the circumference P.



We then see that the area of PQR equals

$$\underbrace{2 \times \frac{1}{2} \times 1^2 \times \sin QOP}_{\text{triangles}} + \underbrace{\frac{1}{2} \times 1^2 \times 2\theta}_{\text{sector}} = \sin(\pi - \theta) + \theta = \sin\theta + \theta$$

and **the answer is (b).**

2. (i) [2 marks] Note that

$$\begin{aligned} f^2g(x) = f(f(g(x))) = f(f(2x)) = f(2x+1) = 2x+2; \\ gf(x) = g(x+1) = 2(x+1) = 2x+2. \end{aligned}$$

(ii) [3 marks] Using the identity $f^2g = gf$ we see that

$$gf^2g = g(f^2g) = g(gf) = g^2f$$

and also that

$$gf^2g = gf(fg) = f^2g(fg) = f^2gfg$$

and finally that

$$f^2gfg = f^2(gf)g = f^2\left(f^2g\right)g = f^4g^2.$$

These four, f^4g^2, f^2gfg, gf^2g, g^2f are the only sequences that lead to $4x+4$.

(iii) [4 marks] Note that

$$\begin{aligned} f^k(x) &= x+k; \\ gf^k(x) &= 2(x+k) = 2x+2k; \\ f^jgf^k(x) &= (2x+2k)+j = 2x+2k+j; \\ gf^jgf^k(x) &= 2(2x+2k+j) = 4x+4k+2j; \\ f^igf^jgf^k(x) &= (4x+4k+2j)+i = 4x+4k+2j+i. \end{aligned}$$

(iv) [6 marks] We need to consider the ways we can have $4k+2j+i = 4m$ where $i, j, k \geqslant 0$. Clearly k can take the values $0 \leqslant k \leqslant m$ and then we need

$$2j+i = 4(m-k).$$

Then j can take values $0 \leqslant j \leqslant 2(m-k)$ – that is $2m-2k+1$ choices for j (given k). The choice of j then determines i. So the number of possible combinations is

$$\begin{aligned} \sum_{k=0}^{m}(2m-2k+1) &= (2m+1)\left(\sum_{k=0}^{m}1\right) - 2\left(\sum_{k=0}^{m}k\right) \\ &= (2m+1)(m+1) - 2\times\frac{1}{2}m(m+1) \\ &= (m+1)(m+1) \\ &= (m+1)^2. \end{aligned}$$

3. (i) [2 marks] For there to be two distinct turning points the derivative $f'(x) = 3x^2 + 2ax + b$ must have two distinct real roots. This is determined by the discriminant

$$(2a)^2 - 4 \times 3 \times b = 4\left(a^2 - 3b\right).$$

Thus $y = f(x)$ has two distinct turning points if, and only if, $a^2 - 3b > 0$.

(ii) [3 marks] The x-coordinates of the turning points are given by

$$x = \frac{-2a \pm \sqrt{4a^2 - 12b}}{2 \times 3} = -\frac{a \pm \sqrt{a^2 - 3b}}{3}.$$

If we call these x_1 and x_2 with $x_1 < x_2$ then

$$\begin{aligned} x_2 - x_1 &= \frac{1}{3}\left(-a + \sqrt{a^2 - 3b} + a + \sqrt{a^2 - 3b}\right) \\ &= \frac{2}{3}\sqrt{a^2 - 3b}. \end{aligned}$$

(iii) [4 marks] The equation of the translated graph has a repeated root at $x = 0$ and another root at t (say), where t is a positive real number. It is thus the case that

$$g(x) = x^2(x - t) = x^3 - tx^2, \qquad \text{and} \qquad g'(x) = 3x^2 - 2tx = x(3x - 2t).$$

There are then turning points at $x = 0$ and $x = \frac{2}{3}t$. Thus, using the result from (ii), we see that

$$\frac{2}{3}t = \frac{2}{3}\sqrt{a^2 - 3b}$$

and hence $t = \sqrt{a^2 - 3b}$, as required.

(iv) [4 marks] The area of the region R is given by

$$\begin{aligned} -\int_0^t g(x)\,\mathrm{d}x &= -\int_0^t x^2(x - t)\,\mathrm{d}x \\ &= -\left[\frac{x^4}{4} - \frac{x^3}{3}t\right]_0^t \\ &= \frac{t^4}{12} \\ &= \frac{\left(a^2 - 3b\right)^2}{12}, \end{aligned}$$

which is rational when the coefficients a, b are rational.

(v) [2 marks] Yes, it is possible for R to be rational when a and b are both irrational. For example, let $a = 2\sqrt[4]{2}$ and $b = \sqrt{2}$.

4. (i) [2 marks] We need the gradient of the line segment joining (x, x^2) to $C = (0, 2)$. This is, for $x \neq 0$, given by

$$\frac{x^2 - 2}{x - 0} = \frac{x^2 - 2}{x}.$$

(ii) [3 marks] Let the coordinates of B be (x_1, x_1^2). Then the gradient of the parabola (and also of the circle) at B is equal to $2x_1$. The tangent to these curves at B is perpendicular to the line segment joining (x_1, x_1^2) to $(0, 2)$, so that

$$\frac{x_1^2 - 2}{x_1} \times 2x_1 = -1.$$

Solving this, and taking the appropriate solution, gives $x_1 = \sqrt{\frac{3}{2}}$ and $x_1^2 = \frac{3}{2}$.
(iii) [4 marks] The square of the radius CB is

$$\left(\sqrt{\frac{3}{2}}\right)^2 + \left(2 - \frac{3}{2}\right)^2 = \frac{7}{4},$$

so the area of the circle is $\frac{7\pi}{4}$. The angle at the centre subtended by the minor arc AB is

$$2\cos^{-1}\left(\frac{\frac{1}{2}}{\sqrt{\frac{7}{4}}}\right) = 2\cos^{-1}\left(\frac{1}{\sqrt{7}}\right),$$

from which we see the sector's area equals

$$\frac{1}{2}r^2\theta = \frac{1}{2} \times \frac{7}{4} \times 2\cos^{-1}\left(\frac{1}{\sqrt{7}}\right) = \frac{7}{4}\cos^{-1}\left(\frac{1}{\sqrt{7}}\right)$$

(iv) [2 marks] Consider the equation

$$\frac{x^2 - a}{x} \times 2x = -1.$$

This can be rearranged to give $x^2 = a - \frac{1}{2}$. For two distinct real solutions we require $a > \frac{1}{2}$.

(v) [4 marks] If $0 < a \leqslant \frac{1}{2}$ then, from (iv), the circle is resting on the vertex of the parabola. In this case $a = r$.

On the other hand, if $a > \frac{1}{2}$ then there are two distinct points points of contact, and we have

$$r^2 = x^2 + (x^2 - a)^2.$$

Now, using the fact that $x^2 = a - \frac{1}{2}$, we obtain

$$r^2 = \left(a - \frac{1}{2}\right) + \frac{1}{4} \implies a = r^2 + \frac{1}{4}.$$

5. (i) [1 mark] From the recursive definition

$$\begin{aligned} P_0 &= \mathbf{F}, \\ P_1 &= P_0\,\mathbf{L}\,P_0\,\mathbf{R} = \mathbf{F\,L\,F\,R} \\ P_2 &= P_1\,\mathbf{L}\,P_1\,\mathbf{R} = \mathbf{F\,L\,F\,R\,L\,F\,L\,F\,R\,R}. \end{aligned}$$

(ii) [1 mark] Say there are f_n commands $\mathbf{F}$ in P_n. As $P_{n+1} = P_n\,\mathbf{L}\,P_n\,\mathbf{R}$ then $f_{n+1} = 2f_n$. As $f_0 = 1$ then $f_n = 2^n$.

(iii) [3 marks] Let l_n denote the total number of commands in P_n. As $P_{n+1} = P_n\,\mathbf{L}\,P_n\,\mathbf{R}$ then

$$l_{n+1} = l_n + 1 + l_n + 1 = 2l_n + 2.$$

If we set $m_n = l_n + 2$ (following the hint) then we see

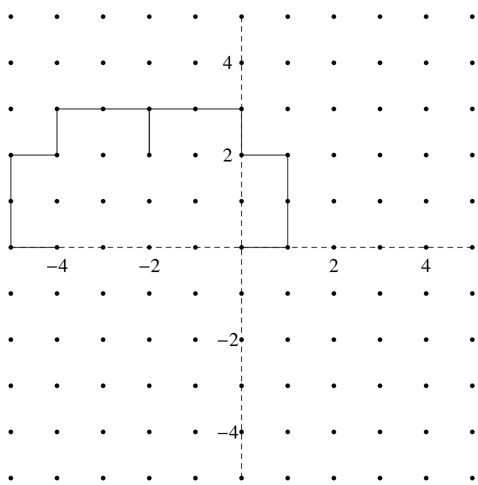
$$m_{n+1} = l_{n+1} + 2 = 2l_n + 4 = 2\,(l_n + 2) = 2m_n.$$

As $m_0 = l_0 + 2 = 3$ then $m_n = 3 \times 2^n$ and

$$l_n = m_n - 2 = 3 \times 2^n - 2.$$

(iv) [1 mark] The robot again faces along the positive x-axis after each P_n because each P_n contains as many **L**s as it does **R**s.

(v) [3 marks] P_4 is sketched below:



(vi) [6 marks] After performing P_n the robot sits at (x_n, y_n) facing "east". Turning left it now faces "north" so what otherwise would have led to a movement of (x_n, y_n) instead achieves $(-y_n, x_n)$. So from the recursion $P_{n+1} = P_n\,\mathbf{L}\,P_n\,\mathbf{R}$ we can see that

$$(x_{n+1}, y_{n+1}) = (x_n, y_n) + (-y_n, x_n) = (x_n - y_n, x_n + y_n)\,.$$

Note then that

$$\begin{aligned} (x_{n+2}, y_{n+2}) &= (x_{n+1} - y_{n+1}, x_{n+1} + y_{n+1}) = (-2y_n, 2x_n)\,; \\ (x_{n+4}, y_{n+4}) &= (-2y_{n+2}, 2x_{n+2}) = (-4x_n, -4y_n)\,; \\ (x_{n+8}, y_{n+8}) &= (-4x_{n+4}, -4y_{n+4}) = (16x_n, 16y_n)\,. \end{aligned}$$

Hence

$$(x_8, y_8) = (16, 0) \qquad \text{and} \qquad (x_{8k}, y_{8k}) = \left(16^k, 0\right).$$

6.

(i) [3 marks] Alice's hat is red, and the others are blue. It must be that Alice can see that neither of the others has a red hat, so can deduce the colour of her own.

(ii) [3 marks] Alice must be able to see a red hat, or would be able to deduce the colour of her own hat. Likewise, Bob must be able to see a red hat, or would be able to deduce the colour of his own hat (given Alice's answer). Hence Charlie's hat is red.

(iii) [3 marks] Alice must be able to see two hats of the same colour in order to deduce the colour of her hat. Bob knows this, and so deduces his hat is the same colour as Charlie's. Hence Alice's hat is blue, and Bob's and Charlie's are red.

(iv) [3 marks] Alice must be able to see two hats of opposite colours, or else she would be able to deduce her own hat colour. Bob knows this, so deduces his hat is a different colour from Charlie's. Hence Charlie's hat is blue.

(v) [3 marks] If Bob and Charlie had different colour hats, Alice would know that she and Bob both had red hats. Therefore Bob and Charlie both have red hats.

7. (i) [1 mark] If Amy plays 1, Brian plays 2 and wins; if Amy plays 2, Brian plays 1 and wins.

(ii) [2 marks] If Amy starts with 0, Brian can then play 1. Amy is now permitted to play 0 or 2. If Amy plays 0, Brian plays 2 or if Amy plays 2, Brian plays 0. Either way Brian wins after two rounds.

(iii) [1 mark] If Amy plays 0, Brian plays 2 to win; if Amy plays 2, Brian plays 0 to win.

(iv) [3 marks] Brian playing 2 would effectively return the game to the starting position having used up one turn. Amy will continue by playing 1. If this leads to a win for Brian following the sequence | 1, 2 | 1, then w, then Brian could have won more quickly by following 1 then w.

(v) [1 mark] After | 1, 0 | if Amy now plays 1 then Brian can win by playing 0. Hence Amy should play 2.

(vi) [7 marks] Following | 1, 0 | 2, if Brian plays 0, then we're effectively back in the starting position. So Brian should play 1.

After | 1, 0 | 2, 1 | if Amy plays 1, Brian can win with 0. Hence Amy should play 2.

We're now in a cycle, leading to | 1, 0 | 2, 1 | 2, 1 | 2, 1 | 2, 1 | giving a win for Amy.