THE COLLEGES OF OXFORD UNIVERSITY

MATHEMATICS

SUNDAY 12 DECEMBER 2004

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

Write your name, college (where you are sitting the test), and your proposed course (from the list above) in BLOCK CAPITALS

NAME:

COLLEGE:

COURSE:

Attempt all the questions. Each of the ten parts of Question 1 is worth 4 marks, and Questions 2,3,4,5 are worth 15 marks each, giving a total of 100.

Question 1 is a multiple choice question for which marks are given solely for the correct answers. Answer Question 1 on the grid on Page 2. Write your answers to Questions 2,3,4,5 in the space provided, continuing on the back of this booklet if necessary.

THE USE OF CALCULATORS OR FORMULA SHEETS IS PROHIBITED.

1. For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. Please show any rough working in the space provided between the parts.

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. How many values of x satisfy the equation

$$\sin 2x + \sin^2 x = 1$$

in the range $0 \leq x < 2\pi$?

(a) 2 (b) 4 (c) 6 (d) 8

B. The smallest value of the function

$$f(x) = 2x^3 - 9x^2 + 12x + 3$$

in the range $0 \leq x \leq 2$ is

(a) 1 (b) 3 (c) 5 (d) 7

C. The turning point of the parabola

$$y = x^2 - 2ax + 1$$

is closest to the origin when

(a) $a = 0$ (b) $a = \pm 1$ (c) $a = \pm\frac{1}{\sqrt{2}}$ or $a = 0$ (d) $a = \pm\frac{1}{\sqrt{2}}$.

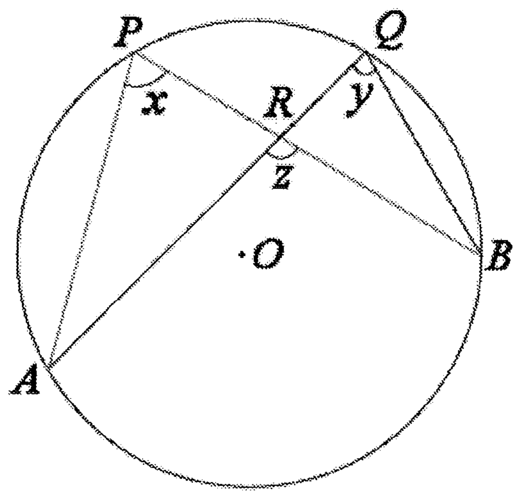
D. What is the reflection of the point $(3, 4)$ in the line $3x + 4y = 50$?

(a) $(9, 12)$ (b) $(6, 8)$ (c) $(12, 16)$ (d) $(16, 12)$

E. Two different faces of a cube are chosen at random. What is the chance of them being opposite one another?

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{5}$ (d) $\frac{1}{6}$

F.



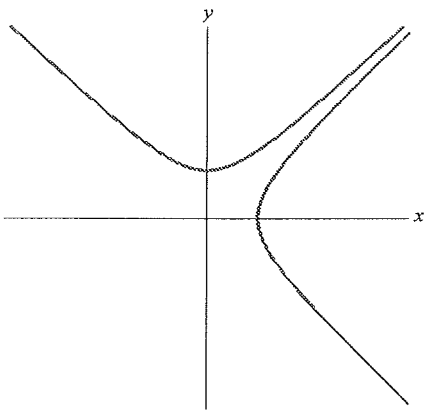
In the figure above, A, B, P and Q are arbitrary distinct points on a circle centred at O. The chords AQ and BP meet at R, and the angles $\measuredangle APB$, $\measuredangle AQB$ and $\measuredangle ARB$ are equal to x, y and z respectively. Which of the following statements is true?

(a) $x + y = \frac{\pi}{2}$;
(b) $x + y = z$;
(c) $x + y = 2x$;
(d) $x + y = \pi$.

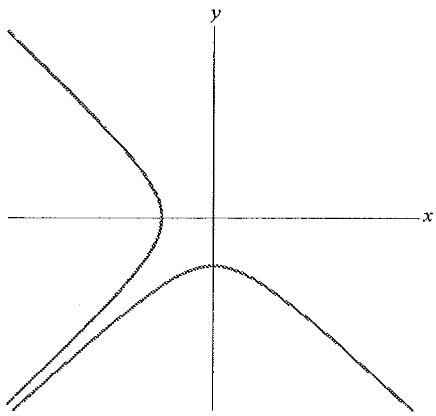
G. Which of the following numbers is largest in value? (All angles are given in radians.)

(a) $\tan\left(\frac{5\pi}{4}\right)$ (b) $\sin^2\left(\frac{5\pi}{4}\right)$ (c) $\log_{10}\left(\frac{5\pi}{4}\right)$ (d) $\ln\left(\frac{5\pi}{4}\right)$

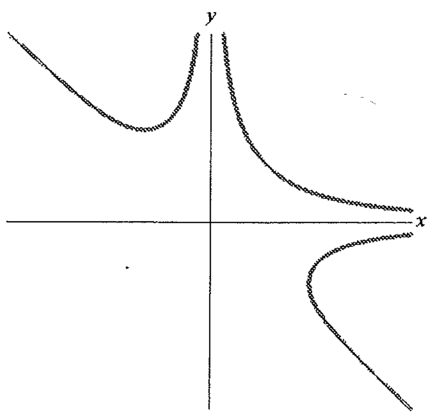
H. A sketch of the curve with equation $x^2y^2(x+y)=1$ is drawn in



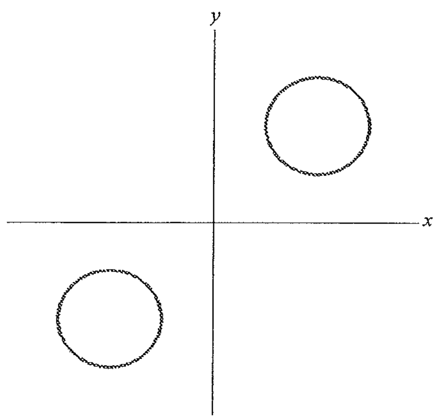
(a)



(b)



(c)



(d)

I. Given numbers a, b, c, which of the following statements about the simultaneous equations

$$\begin{aligned} 2x + y &= 5, \\ ax + by &= c, \end{aligned}$$

is true?

(a) There are no solutions when $a = 2b$ and $c = 5b$;
(b) There is a unique solution when $a \neq 2b$ and $c = 5b$;
(c) There are an infinite number of solutions when $a = 2, b = 1$, and $c = 0$;
(d) There are no solutions when $a \neq 2b$ and $c = 5b$.

J. You are given two whole numbers p and q, and told that three of the following statements concerning p and q are *true* and that the other statement is *false*. Which is the false statement?

(a) pq is even;
(b) $p + q$ is even;
(c) $2p + q^2$ is odd;
(d) $p^2 + 2q$ is odd.

2. (a) For what values of the constant k does the quadratic equation

$$x^2 - 2x - 1 = k$$

have:

(i) no real solutions;
(ii) one real solution;
(iii) two real solutions.

(b) Showing your working, express $(x^2 - 2x - 1)^2$ as a polynomial of degree 4 in x.

(c) Show that the quartic equation

$$x^4 - 4x^3 + 2x^2 + 4x + 1 = h$$

has exactly two real solutions if *either* $h = 0$ *or* $h > 4$.

Show that there is no value of h such that the above quartic equation has just one real solution.

3. Let

$$f(x) = \begin{cases} x+1 & \text{for } 0 \le x \le 1; \\ 2x^2 - 6x + 6 & \text{for } 1 \le x \le 2. \end{cases}$$

(a) On the axes provided below, sketch a graph of $y = f(x)$ for $0 \le x \le 2$, labelling any turning points and the values attained at $x = 0, 1, 2$.

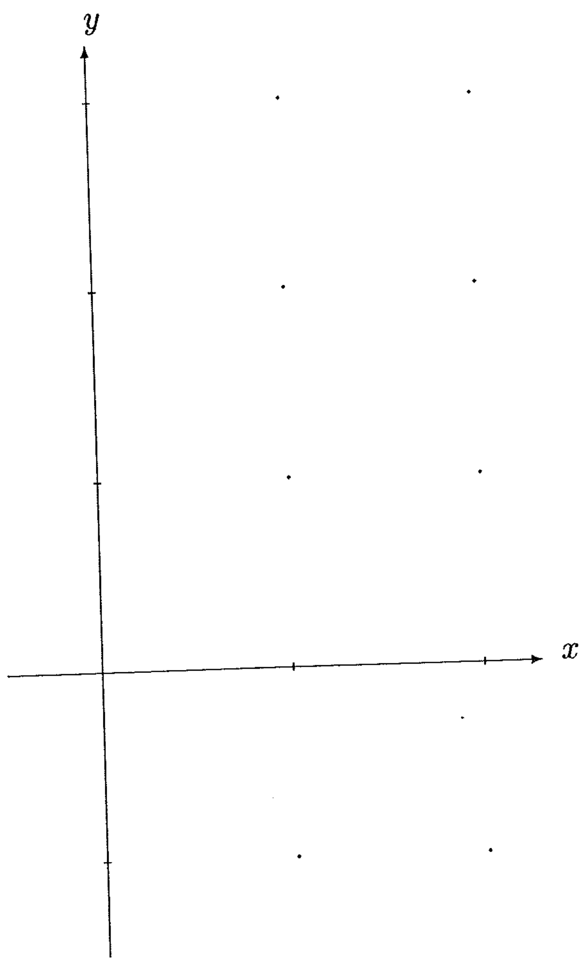
(b) For $1 \le t \le 2$, define

$$g(t) = \int_{t-1}^{t} f(x)\, dx.$$

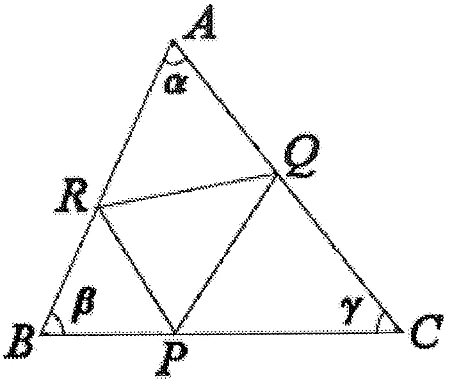
Express $g(t)$ as a cubic in t.

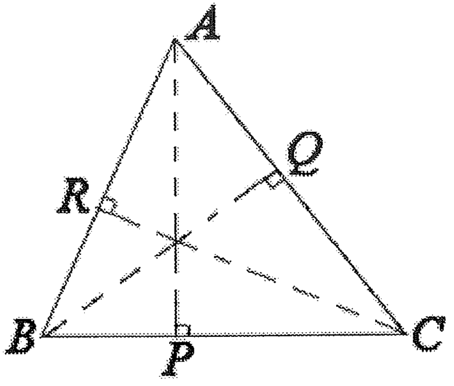
(c) Calculate and factorize $g'(t)$.

(d) What are the minimum and maximum values of $g(t)$ for t in the range $1 \le t \le 2$?



4.





The triangle ABC, drawn above, has sides BC, CA and AB of length a, b and c respectively, and the angles at A, B and C are α, β and γ.

(a) Show that the area of ABC equals $\frac{1}{2}bc\sin\alpha$.

Deduce the sine rule

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

(b) In the triangle above, let P, Q and R respectively be the feet of the perpendiculars from A to BC, B to CA, and C to AB, as shown.

Prove that

$$\text{Area of } PQR = (1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma) \times (\text{Area of } ABC).$$

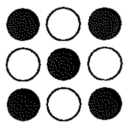
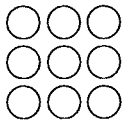
For what triangles ABC, with angles α, β, γ, does the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

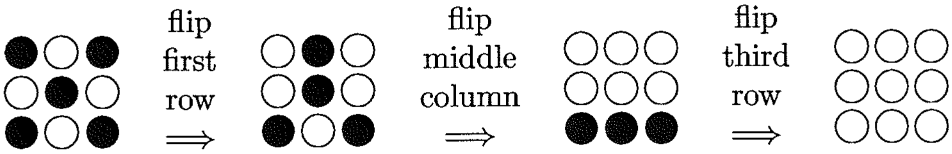
hold?

5. The game of *Oxflip* is for one player and involves circular counters, which are white on one side and black on the other, placed in a grid. During a game, the counters are flipped over (changing between black and white side uppermost) following certain rules.

Given a particular size of grid and a set starting pattern of whites and blacks, the aim of the game is to reach a certain target pattern. Each "move" of the game is to flip over either a whole row or a whole column of counters (so one whole row or column has all its blacks swapped to whites and vice versa). For example, in a game played in a three-by-three square grid, if you are given the starting and target patterns

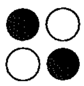
Start:  Target: 

a sequences of three moves to achieve the target is:



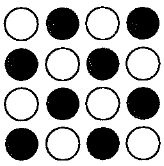
There are many other sequences of moves which also have the same result.

(a) Consider the two-by-two version of the game with starting pattern



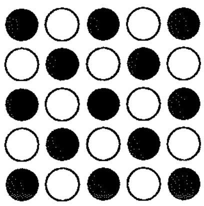
Draw, in the blank patterns opposite, the eight different target patterns (including the starting pattern) which it is possible to obtain. What are the possible numbers of white counters that may be present in these target patterns?

(b) In the four-by-four version of the game, starting with pattern



explain why it is impossible to reach a pattern with only one white counter.
[*Hint: don't try to write out every possible combination of moves.*]

(c) In the five-by-five game, explain why any sequence of moves which begins



and ends with an all-white pattern, must involve an odd number of moves. What is the least number of moves needed? Give reasons for your answer.