THE COLLEGES OF OXFORD UNIVERSITY

MATHEMATICS

SUNDAY 14 DECEMBER 2003

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

Write your name, college (where you are sitting the test), and your proposed course (from the list above) in BLOCK CAPITALS

NAME:

COLLEGE:

COURSE:

Attempt all the questions. Each of the ten parts of Question 1 is worth 4 marks, and and Questions 2,3,4,5 are worth 15 marks each, giving a total of 100.

Question 1 is a multiple choice question for which marks are given solely for the correct answers. Answer Question 1 on the grid on Page 2. Write your answers to Questions 2,3,4,5 in the space provided, continuing on the back of this booklet if necessary.

THE USE OF CALCULATORS OR FORMULA SHEETS IS PROHIBITED.

1. For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. You may use the spaces between the parts for rough working.

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. Depending on the value of the constant d, the equation

$$dx^2 - (d-1)x + d = 0$$

may have two real solutions, one real solution or no real solutions. For how many values of d does it have *just one* real solution?

(a) for one value of d ;

(b) for two values of d ;

(c) for three values of d ;

(d) for infinitely many values of d .

B. You are given that e^3 is approximately 20 and that 2^{10} is approximately 1000. *Using this information,* a student can obtain an approximate value for $\ln 2$. Which of the following is it?

(a) $\frac{7}{10}$ (b) $\frac{9}{13}$ (c) $\frac{38}{55}$ (d) $\frac{41}{59}$

C. How many solutions does the equation

$$\sin 2x = \cos x$$

have in the range $0 \le x \le \pi$?

(a) one solution;

(b) two solutions;

(c) three solutions;

(d) four solutions.

D. What is the value of the definite integral

$$\int_1^2 \frac{dx}{x + x^3}?$$

(a) $\ln 2 - \pi/6$

(b) $2\ln 2 - \ln 5$

(c) $\frac{1}{2}\ln\frac{8}{5}$

(d) None of the above.

E. For which real numbers x does the inequality

$$\frac{x}{x^2+1} \leq \frac{1}{2}$$

hold?

(a) for *all* real numbers x;

(b) for real numbers $x \leq \frac{1}{2}$ and no others;

(c) for real numbers $x \leq 1$ and no others;

(d) none of the above.

F. Two players take turns to throw a fair six-sided die until one of them scores a six. What is the probability that the first player to throw the die is the first to score a six?

(a) $\frac{5}{9}$ (b) $\frac{3}{5}$ (c) $\frac{6}{11}$ (d) $\frac{7}{12}$

G. For which of the following do we have

$$\frac{dy}{dx} = 2y \ln y?$$

(a) $y = e^{e^{2x}}$

(b) $y = e^{2e^{x}}$

(c) $y = e^{e^{x^2}}$

(d) $y = 2e^{e^{x}}$

H. Into how many regions is the plane divided when the following three parabolas are drawn?

$$y = x^2$$
$$y = x^2 - 2x$$
$$y = x^2 + 2x + 2$$

(a) 4 (b) 5 (c) 6 (d) 7

I. You go into a supermarket to buy two packets of biscuits, which may or may not be of the same variety. The supermarket has 20 different varieties of biscuits and at least two packets of each variety. In how many ways can you choose your two packets?

(a) 400 (b) 210 (c) 200 (d) 190

J. There are real numbers x, y such that precisely *one* of the statements (a), (b),(c), (d) is true. Which is the true statement?

(a) $x \geq 0$

(b) $x < y$

(c) $x^2 > y^2$

(d) $|x| \leq |y|$

Turn Over

2. Let k and n be positive integers such that $n \geq 2k$ and

$$\frac{(n-2)!}{(n-2k)!} = k!2^{k-1}.$$

(Recall that, for $r \geq 1$, $r!$ is the product $r.(r-1).(r-2)\dots 2.1$ and that $0!$ is defined to equal 1.)

(i) Suppose that $k = 1$. What are the possible values of n?

(ii) Suppose that $k = 2$. Show that $(n-2)(n-3) = 4$. What are the possible values of n?

(iii) Suppose that $k = 3$. Show that it is impossible that $n \geq 7$.

(iv) Suppose that $k \geq 4$. Show that there are no possible values of n.

3. Let O, P, P_1, P_2 be the points in the (x, y)-plane with coordinates $(0, 0)$, $(s, 1/s)$, $(s_1, 1/s_1)$, $(s_2, 1/s_2)$ respectively.

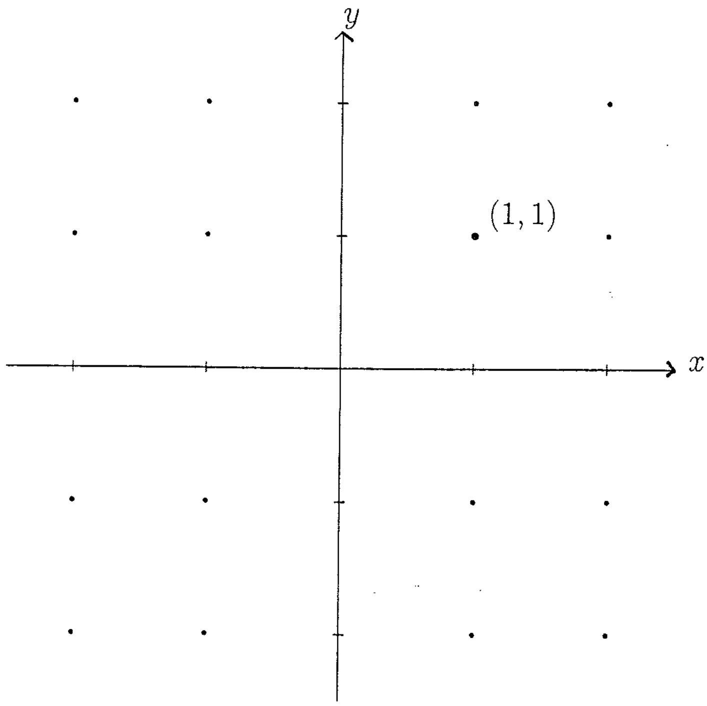
(i) Using the axes below, sketch the curve traced out by P as s varies over non-zero real values, and find an equation for the curve in the form $y = f(x)$.

(ii) Write down the equation of the straight line PP_1 joining P to P_1, giving your answer in the form $y = m_1x + c_1$.

(iii) Show that the line PP_1 is perpendicular to PP_2 if, and only if, $s_1s_2 = -1/s^2$.

(iv) Let m_1, m_2, n_1, n_2 be the gradients of the lines PP_1, PP_2, OP_1, OP_2 respectively. Show that

$$\left(\frac{m_1}{m_2}\right)^2 = \frac{n_1}{n_2}$$



4. In this question we shall consider the function $f(x)$ defined by

$$f(x) = x^2 - 2px + 3,$$

where p is a constant.

(i) Show that the function $f(x)$ has one stationary value in the range $0 < x < 1$ if $0 < p < 1$, and no stationary values in that range otherwise.

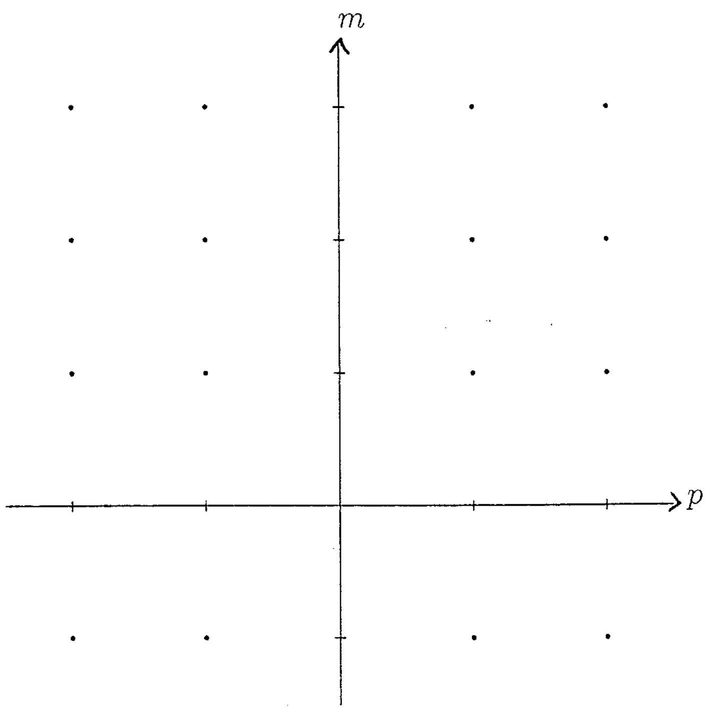
In the remainder of the question, we shall be interested in the *smallest* value attained by $f(x)$ in the range $0 \leq x \leq 1$. Of course, this value, which we shall call m, will depend on p.

(ii) Show that if $p \geq 1$ then $m = 4 - 2p$.

(iii) What is the value of m if $p \leq 0$?

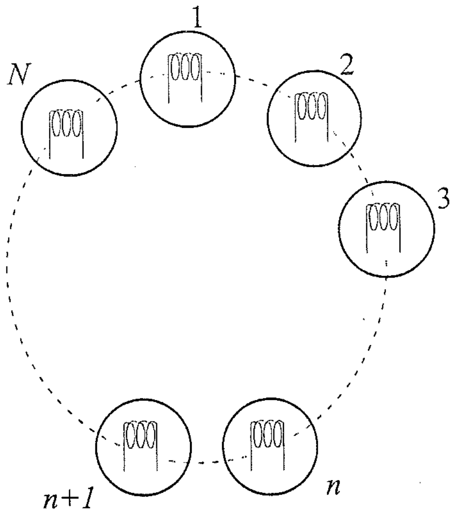
(iv) Obtain a formula for m in terms of p, valid for $0 < p < 1$.

(v) Using the axes opposite, sketch the graph of m as a function of p in the range $-2 \leq p \leq 2$.



Turn Over

5.



The diagram represents an array of N electric lights arranged in a circle. Initially, each light may be set to be ON or OFF in an arbitrary way. After one second the settings are updated according to the following rule which determine the new state of a bulb in terms of the initial states of that bulb and the one just next to it in the clockwise direction.

if initially bulb n and bulb $n+1$ are in the *same* state (i.e. either both OFF or both ON) then after 1 second bulb n will be OFF;

if initially bulb n and bulb $n+1$ are in *different* states (i.e. one OFF the other ON) then after 1 second bulb n will be ON;

Of course, if $n = N$, we replace $n+1$ with 1 in the above.

Subsequently, the settings are updated each second by reapplying the same rule.

(i) Explain why after one second there cannot be exactly one bulb ON.

(ii) More generally, explain why after one second there cannot be an *odd* number of bulbs ON.

(iii) Show that the state of bulb n after 2 seconds is completely determined by the initial states of bulbs n and $n+2$ (with appropriate modifications when $n = N$ or $n = N-1$).

(iv) The initial states of which bulbs determine the state of bulb n after 4 seconds?

(v) Show that if $N = 8$ then, irrespective of the initial settings, all bulbs will eventually be OFF. How long will this take?