Serial Number 32
THE COLLEGES OF OXFORD UNIVERSITY
Entrance Examination in Mathematics

MATHEMATICS I

16 November 1992 Afternoon
Time allowed: 3 hours

Answers to each of Sections A, B and C must be attached to separate cover sheets and handed in separately. If no questions are attempted in any one section the cover sheet should still be handed in. Each cover sheet should be clearly labelled A, B or C.

All candidates must attempt Question 1 *which carries twice the mark for any other question. There is no restriction on the number of questions any candidate may attempt but only Question 1 and the best three solutions to Questions 2-11 will contribute to the total mark for this paper.*

SECTION A

A1. (i) Differentiate $\sin(a\sin^{-1}x)$ with respect to x, where a is a constant.

(ii) Using the identity

$$\frac{(b-a)}{(x+a)(x+b)} = \frac{1}{(x+a)} - \frac{1}{(x+b)},$$

or otherwise, express

$$\frac{1}{(x+a)^2(x+b)^2}$$

in partial fractions for $a \neq b$.

(iii) Evaluate the integral $\displaystyle\int_0^4 \frac{dx}{1+\sqrt{x}}$.

(iv) Evaluate the integral $\displaystyle\int_0^1 x^2 e^{-x} dx$.

(v) Sketch the graph of $y^2 = |x^2 - 9| - 7$.

(vi) By expressing $2\cos 2rt \sin t$ as a difference of sines, prove that

$$1 + 2\sum_{r=1}^{n} \cos 2rt = \frac{\sin(2n+1)t}{\sin t}.$$

Hence find

$$\sum_{r=0}^{n} \cos^2 rt.$$

(vii) Find the minimum distance between the point $(0,1)$ and the curve $y = x^2$.

(viii) Let $f(x) = 2x + |x-1|$.
(a) Sketch the graph of f.
(b) Sketch the graph of the inverse function, f^{-1}.

(ix) Given that $\sqrt{15} = 3.8729833462074168852$ to 19 decimal places evaluate

$$\frac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{2(5-\sqrt{5})(1+\sqrt{3})}$$

to 20 decimal places.

(x) Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Find vectors $\mathbf{p}$ and $\mathbf{q}$ such that $\mathbf{a} = \mathbf{p} + \mathbf{q}$ with $\mathbf{p}$ parallel to $\mathbf{b}$ and $\mathbf{q}$ perpendicular to $\mathbf{b}$.

SECTION B

B2. Consider the following set of linear equations:

$$\begin{array}{rrrcl} 4x & -y & -z & = & 21 \\ 2x & +4y & +z & = & 69 \\ 8x & +y & -z & = & 81 \\ -4x & +7y & +3z & = & 57 \end{array}$$

(a) Find all real solutions (x, y, z) of this set of equations.

(b) How many solutions are there if we also require x, y, z to be positive integers?

B3. (a) Write down a formula connecting $\cos\theta$ and $\cos 2\theta$.

(b) What is the sign of $\cos\theta$ when $0 < \theta < \frac{\pi}{2}$?

(c) Given that $2\cos\frac{\pi}{4} = \sqrt{2}$ prove that $2\cos\frac{\pi}{8} = \sqrt{2+\sqrt{2}}$.

(d) Find a polynomial $f(x)$ of degree 4 with integer coefficients which has $2\cos\frac{\pi}{8}$ as one of its roots.

(e) Write down all the roots of $f(x)$.

(f) For which angles ϕ is $2\cos\phi$ a root of $f(x) = 0$?

(g) Express each root of $f(x) = 0$ in the form $2\cos\phi$.

B4. (a) Write down an expression for the sum of the series $\sum_{r=0}^{\infty}(\alpha x)^r$ where $|\alpha x| < 1$.

Let α, β, γ be three distinct non-zero numbers, and let

$$f(x) = (1-\alpha x)(1-\beta x)(1-\gamma x) = 1 + Ax + Bx^2 + Cx^3.$$

(b) Noting that $C \neq 0$, prove that $(A + 2Bx + 3Cx^2)/(1 + Ax + Bx^2 + Cx^3)$ is not a polynomial of any finite degree m.

(c) By differentiating $\ln f(x)$ prove that

$$\frac{A + 2Bx + 3Cx^2}{1 + Ax + Bx^2 + Cx^3} = \frac{-\alpha}{1-\alpha x} + \frac{-\beta}{1-\beta x} + \frac{-\gamma}{1-\gamma x}.$$

For each natural number n let $s_n = \alpha^n + \beta^n + \gamma^n$.

(d) For $r \geq 2$ prove that $s_{r+1} = -As_r - Bs_{r-1} - Cs_{r-2}$.

(e) Prove that we cannot have $s_{193} = s_{194} = s_{195} = 0$.

B5. For integers n and r let $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ if $0 \le r \le n$, and $= 0$ otherwise.

(a) Write down the coefficient of x^r in $(1-x^2)^m$, where m is a positive integer, distinguishing between the cases where r is even and r is odd.

(b) Noting that $(1-x)(1+x) = 1-x^2$ prove that if t is a positive integer then

(i) $\displaystyle\sum_{k=0}^{2t}(-1)^k\binom{m}{k}\binom{m}{2t-k} = (-1)^t\binom{m}{t}$;

(ii) $\displaystyle\sum_{k=0}^{2t+1}(-1)^k\binom{m}{k}\binom{m}{2t+1-k} = 0.$

(c) By considering $(1+x)^2/x$ raised to a suitable power prove that if t is a positive integer then

$$\sum_{k=0}^{m}\binom{m}{k}\binom{m}{k-t} = \binom{2m}{m+t};$$

it may help you to recall that $\binom{n}{r} = 0$ when $r < 0$.

B6. The Bonnyhill synthesisers (seldom closely encountered) come in a variety of models. The n-th model in the range, the BH-n, is equipped with $n+1$ red buttons labelled with the names of $n+1$ distinct notes $A_0, A_1, \ldots, A_n$ and with n blue buttons labelled 1 to n. To operate the synthesiser one must first press each of the red buttons in some order; the synthesiser then plays over the chosen sequence of $n+1$ notes. Thereafter, whenever a blue button, r say, is pressed the synthesiser plays whatever sequence of $n+1$ notes as it has just played, except it swaps A_{r-1} and A_r. If, however, a blue button is pressed so that the synthesiser plays the sequence $A_0, A_1, \ldots, A_n$, then the synthesiser vanishes completely.

(a) Show that whatever initial sequence is chosen the BH-2 can always be made to vanish after at most 3 blue keypresses.

(b) By induction on n show that the BH-n can always be made to vanish.

(c) Prove that if starting with sequence $\mathcal{A}$ it is possible to play sequence $\mathcal{A}'$ (before the synthesiser vanishes), then it is possible to play $\mathcal{A}$ starting with $\mathcal{A}'$ provided $\mathcal{A}'$ is not $A_0, A_1, \ldots, A_n$.

(d) Prove that the BH-2 can be made to play A_0, A_2, A_1 starting with A_1, A_0, A_2.

(e) What possibilities are there for the sequence of notes played just before $A_0, A_1, \ldots, A_n$ can be played?

(f) Prove that it is possible, starting with any $\mathcal{A}$, not itself $A_0, A_1, \ldots, A_n$, to play any $\mathcal{A}'$.

SECTION C

C7. The function $z(x)$ is defined in terms of x and $y(x)$ by

$$z = \frac{y-x}{y+x}.$$

Show that

$$\frac{dz}{dx} = \frac{2}{(x+y)^2}\left(x\frac{dy}{dx} - y\right).$$

Hence show that the differential equation

$$f(x)\left(x\frac{dy}{dx} - y\right) = (y^2 - x^2)$$

can be transformed into the equation

$$f(x)\frac{dz}{dx} = 2z.$$

Solve the equation

$$(x^3 + x^2 + x + 1)\left(x\frac{dy}{dx} - y\right) = (x-1)(y^2 - x^2)$$

given $y = 3$ when $x = 1$.

C8. For n a non-negative integer, define

$$I_n = \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2nx}{\sin 2x}\, dx.$$

(a) Evaluate I_1.

(b) Show that, for $r \geq 1$,

$$I_{2r+1} - I_{2r-1} = \frac{1-(-1)^r}{2r},$$

$$I_{2r} - I_{2r-2} = \frac{1}{2r-1}.$$

(c) Evaluate I_8 and I_9.

C9. By integrating by parts twice, show that

$$\frac{1}{2}\int_{n}^{n+1}(x-n)(n+1-x)f''(x)\,\mathrm{d}x = \frac{1}{2}\{f(n)+f(n+1)\} - \int_{n}^{n+1} f(x)\,\mathrm{d}x.$$

Given that

$$0 \le (x-n)(n+1-x) \le \frac{1}{4} \qquad \text{for} \quad n \le x \le n+1,$$

show that if $f''(x) \ge 0$ for $n \le x \le n+1$ then

$$0 \le \frac{1}{2}\{f(n)+f(n+1)\} - \int_{n}^{n+1} f(x)\,\mathrm{d}x \le \frac{1}{8}\{f'(n+1)-f'(n)\}.$$

Deduce that if N is a positive integer such that $f''(x) \ge 0$ for $1 \le x \le N$ then

$$0 \le \sum_{n=1}^{N} f(n) - \frac{1}{2}f(1) - \frac{1}{2}f(N) - \int_{1}^{N} f(x)\,\mathrm{d}x \le \frac{1}{8}\{f'(N)-f'(1)\}.$$

By taking $f(x) = \dfrac{1}{x}$, or otherwise, prove that for all positive integers N

$$0 \le \sum_{n=1}^{N}\frac{1}{n} - \ln N \le \frac{5}{8} + \frac{1}{2N} - \frac{1}{8N^2}.$$

C10. Prove that

$$\frac{d^n}{dx^n}(1-x)^{-\frac{1}{2}} = \tfrac{1}{2}\cdot\tfrac{3}{2}\cdots(n-\tfrac{1}{2})\,(1-x)^{-n-\frac{1}{2}}.$$

Hence, or otherwise, show that for $-1 < x < 1$,

$$(1-x)^{-\frac{1}{2}} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

where $a_0 = 1$ and $a_{n+1} = \dfrac{n+\frac{1}{2}}{n+1}a_n$, $n = 0,1,2,\ldots$. Show that $a_n > 0$ for all n and $a_0 > a_1 > a_2 > \cdots$. Hence prove that, for $0 < x < 1$,

$$0 < a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots < a_{n+1}(x^{n+1} + x^{n+2} + \cdots),$$

and so

$$0 < (1-x)^{-\frac{1}{2}} - (a_0 + a_1x + \cdots + a_nx^n) < a_{n+1}\frac{x^{n+1}}{1-x}.$$

Deduce that the error in approximating $\sqrt{2}$ by using the identity

$$\sqrt{2} = \frac{7}{5}\left(1 - \frac{1}{50}\right)^{-\frac{1}{2}},$$

and taking $n = 3$ is less than $6{\cdot}25 \times 10^{-8}$.

C11. Find any turning points of the function y defined by

$$y = \frac{(x+a)(x+b)}{(x-a)(x-b)}, \qquad a+b \neq 0, \qquad ab \neq 0,$$

distinguishing between the cases $ab > 0$ and $ab < 0$.

Plot y in each of the following cases, carefully determining and marking any turning points, asymptotes and zeros:

(a) $0 < a < b$,

(b) $-b < a < 0 < b$.