OCR Single Pure Revision Sheet

The OCR single maths A level is examined with three compulsory 2 hour exams. Each paper carries equal weight ($33\frac{1}{3}\%$) and each paper is marked out of 100 marks.

Paper 1 contains only pure mathematics.
Paper 2 contains pure mathematics and statistics (roughly 50 marks for each section).
Paper 3 contains pure mathematics and mechanics (roughly 50 marks for each section).

Therefore your A level is roughly $\frac{2}{3}$ pure maths, $\frac{1}{6}$ statistics, and $\frac{1}{6}$ mechanics. This revision sheet *should* cover all of the pure maths you need. *Please* get in contact if you spot anything missing.

I hope you find this revision sheet useful and wish you the very best of luck with your studies.

J.M.S.

Preliminaries

- Changing the subject of an equation. For example in $y = \sqrt{x+6}$, y is the subject of the equation. To change the subject to x we merely need to re–arrange to $x = y^2 - 6$. A harder example is make to make g the subject of $T = 2\pi\sqrt{\frac{l}{g}}$.

$$2\pi\sqrt{\frac{l}{g}} = T \qquad \Rightarrow \qquad \frac{l}{g} = \left(\frac{T}{2\pi}\right)^2 \qquad \Rightarrow \qquad g = l\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 l}{T^2}.$$

- To solve simultaneous equations, isolate x or y from one of the equations and substitute into the other. For example, solve

$$x + 2y = 1,$$
$$x^2 - 2y^2 = 31.$$

  From the first we find $x = 1 - 2y$ and putting into the second we find $(1-2y)^2 - 2y^2 = 31$, which simplifies to $y^2 - 2y - 15 = 0$. This gives $y = -3$ or $y = 5$. To calculate the x values we put the y solutions into either original equation. The solutions are $(-9, 5)$ and $(7, -3)$. [Give your solutions as coordinates to show which x and y values go together.]

- If you ever need to find where two lines or curves cross, then merely view it as a pair of simultaneous equations to be solved.

- You must also know how to handle algebraic fractions and how to write two algebraic fractional expressions as one fraction. The general rules are

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

  Therefore to write $x - \frac{x}{x+1}$ as a single fraction we do the following

$$x - \frac{x}{x+1} = \frac{x}{1} - \frac{x}{x+1} = \frac{x(x+1) - x}{x+1} = \frac{x^2}{x+1}.$$

- *Any* line or curve crosses the x-axis when $y = 0$. Similarly, any line or curve crosses the y-axis when $x = 0$. So to find where $y = x^2 + x - 12$ crosses the x-axis we solve $0 = x^2 + x - 12$ and find $(3, 0)$ and $(-4, 0)$. To find where it crosses the y-axis we put in $x = 0$ to discover $(0, -12)$.

- Always, always, always draw a sketch in any problem that is even vaguely geometric; the sooner you do, the sooner you'll get full marks.

Coordinates, Points and Lines

- Mid point of (x_1, y_1), (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. Average the x-coordinates and average the y-coordinates.
- Distance from (x_1, y_1) to (x_2, y_2) is (by Pythagoras) $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Be careful about negatives! Remember $(2 - (-3))^2 = (2 + 3)^2$.
- Gradient is defined to be
$$\frac{\text{difference in } y}{\text{difference in } x} = \frac{y_2 - y_1}{x_2 - x_1}.$$
If you need the gradient between two points you should visualise them first to see if you should be getting a positive or negative answer. This should also give you an idea of whether to expect a big (steep) of small (shallow) gradient.
- Two lines with gradients m_1 and m_2 are at right angles (perpendicular) if $m_1 \times m_2 = -1$. So if a line has gradient -3 then the line perpendicular to it has gradient $\frac{1}{3}$.
- Lines can be written in many forms, the most common being $y = mx + c$ and $ax + by = c$. Any form can be converted to any other. For example write $3x - 2y = 4$ in the form $y = mx + c$.
$$\begin{aligned} 3x - 2y &= 4 \\ 2y &= 3x - 4 \\ y &= \tfrac{3}{2}x - 2. \end{aligned}$$
- Given one point (x_1, y_1) and a gradient m the line is given by $y - y_1 = m(x - x_1)$.

Surds

- Know and understand the laws
$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b} \qquad \text{and} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$
In particular know how to deal with $\frac{\sqrt{44}}{2}$; it is *not* $\sqrt{22}$! It is $\frac{\sqrt{44}}{2} = \frac{\sqrt{44}}{\sqrt{4}} = \sqrt{\frac{44}{4}} = \sqrt{11}$. This comes up in solving quadratics by the formula; check that when you solve $x^2 + 4x - 2 = 0$ by the formula you obtain $x = -2 \pm \sqrt{6}$.
- You also need to be able to rationalise the denominator of certain types of surd expressions. For example to rationalise $\frac{9}{\sqrt{3}}$ is easy; just multiply by $\frac{\sqrt{3}}{\sqrt{3}}$ to obtain $\frac{9\sqrt{3}}{3} = 3\sqrt{3}$. In harder examples you must multiply the top and bottom of the fraction by the denominator with the sign 'flipped'. For example
$$\frac{2 + 2\sqrt{3}}{5 - 2\sqrt{3}} = \frac{2 + 2\sqrt{3}}{5 - 2\sqrt{3}} \times \boxed{\frac{5 + 2\sqrt{3}}{5 + 2\sqrt{3}}} = \frac{10 + 4\sqrt{3} + 10\sqrt{3} + 12}{25 + 10\sqrt{3} - 10\sqrt{3} - 12} = \frac{22 + 14\sqrt{3}}{13}.$$

Some Important Graphs

- Know the shape of the graph $y = x^n$ for $n = \{1, 2, 3, 4 \ldots\}$.

- If the power is even, then the graph will be $\bigcup$-shaped. They all pass through the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$. The bigger the power, the faster it goes to infinity. Slightly more subtle is the point that in the range $-1 < x < 1$ then the higher the power, the *smaller* y-value (because $0.2\times0.2\times0.2 < 0.2\times0.2$). They are all 'even functions' with the y-axis as a line of symmetry.

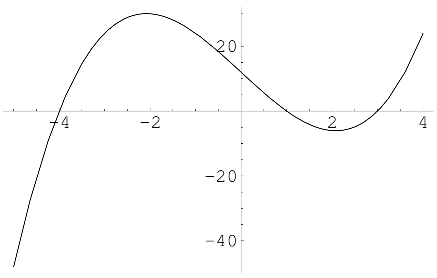
- If the power is odd then they will (with the exception of $y = x^1 = x$, which is a straight line) be shaped like $y = x^3$. They all pass through $(-1, -1)$, $(0, 0)$ and $(1, 1)$. Similar arguments as for even powers exist here. They are all 'odd functions' with the origin being a point of rotational symmetry.

- The family of curves $y = ax^2 + bx + c$ are parabolas. If a is positive then you get a "happy" $\bigcup$-type curve. If a is negative then you get a "sad" $\bigcap$-type curve. They have a line of symmetry and a vertex (turning point) that you can discover by completing the square (see later).

- If you have a curve that is factorised then you can sketch it easily. For example

$$y = (x - 1)(x + 4)(x - 3)$$

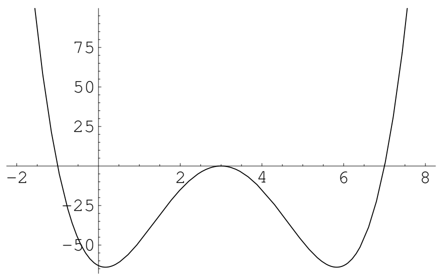
  is a cubic curve that crosses the x-axis at $(1, 0)$, $(-4, 0)$ and $(3, 0)$. It crosses the y-axis when $x = 0$, which gives $(0, 12)$. If x is huge, y is huge and positive and if x is massively negative, then so is y. So



- If a factor is repeated, then it merely touches the x-axis at that point. So

$$y = (x - 3)^2(x + 1)(x - 7)$$

  is a quartic curve that crosses the x-axis at $(-1, 0)$ and $(7, 0)$, but only touches at $(3, 0)$.



Quadratics

- Factorising quadratics. To check whether a given quadratic factorises calculate the discriminant $b^2 - 4ac$; if it is a perfect square (4, 49, 81 etc.) then it factorises.

- When the x^2 coefficient (the number in front of the x^2) is one this is easy. Just spot two numbers which multiply to the constant and add to the x coefficient. For example with $x^2 + 8x + 15$ we need to find two numbers which multiply to 15 and sum to 8; clearly 3 and 5. So $x^2 + 8x + 15 = (x + 3)(x + 5)$.

- If the x^2 coefficient is not one then more work is required. You need to multiply the x^2 coefficient by the constant term and then find 2 numbers which multiply to this and sum to the x coefficient. For example with $6x^2+x-12$ we calculate $6\times-12 = -72$ so the two numbers are clearly 9 and -8. So

$$\begin{aligned} 6x^2 + x - 12 &= 6x^2 + 9x - 8x - 12 & &= 6x^2 - 8x + 9x - 12 \\ &= 3x(2x + 3) - 4(2x + 3) & &= 2x(3x - 4) + 3(3x - 4) \\ &= (3x - 4)(2x + 3) & &= (2x + 3)(3x - 4). \end{aligned}$$

  Notice that it does not matter which way round we write the $9x$ and $-8x$.

- For quadratics that cannot be factorised we need to use the formula. For $ax^2 + bx + c = 0$ the solution is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- The $b^2 - 4ac$ part is called the *discriminant*. If it is positive then there are two *distinct* roots. If it is zero then there exists only one root and it is *repeated*. If it is negative then there are no roots. For example: find the values of k such that $x^2 + (k + 3)x + 4k = 0$ has only one root. We need the discriminant to be zero, so

$$\begin{aligned} b^2 - 4ac &= 0 \\ (k + 3)^2 - 16k &= 0 \\ k^2 - 10k + 9 &= 0 \\ k = 9 &\text{ or } k = 1. \end{aligned}$$

- Completing the square. All about halving the x coefficient into the bracket and then correcting the constant term. For example $x^2 - 6x + 10 = (x - 3)^2 - 9 + 10 = (x - 3)^2 + 1$. If the x^2 coefficient isn't one then need to factorise it out. For example

$$\begin{aligned} -2x^2 + 4x - 8 &= -2[x^2 - 2x] - 8 \\ &= -2[(x - 1)^2 - 1] - 8 \\ &= -2(x - 1)^2 - 6. \end{aligned}$$

  From this we can find the maximum or minimum of the quadratic. For $y = -2(x - 1)^2 - 6$ it is when $x = 1$ (to make the bracket 0) and therefore $y = -6$. In this case $(1, -6)$ is a maximum due to negative x^2 coefficient.

  We can also find the vertical line of symmetry by completing the square. For example

$$\begin{aligned} 3x^2 + 5x + 1 &= 3[x^2 + \tfrac{5}{3}x] + 1 \\ &= 3[(x + \tfrac{5}{6})^2 - \tfrac{25}{36}] + 1 \\ &= 3(x + \tfrac{5}{6})^2 - \tfrac{25}{12} + \tfrac{12}{12} \\ &= 3(x + \tfrac{5}{6})^2 - \tfrac{13}{12}. \end{aligned}$$

  From this we see that the vertex is at $(-\frac{5}{6}, -\frac{13}{12})$ and consequently the line of symmetry is $x = -\frac{5}{6}$.

- You must be on the lookout for *quadratics in disguise*. You spot these when there are two powers on the variable and one is *twice* the other (or can be manipulated into such an equation[1]). Most

[1] For example $3x^3 = 5 + \frac{2}{x^3}$ can be manipulated into $3x^6 - 5x^3 - 2 = 0$ where one power is twice the other.

students like to solve these by means of a substitution (although some students don't need to do this). For example to solve $x^4 + 2x^2 = 8$ work as follows:

$$\begin{aligned} x^4 + 2x^2 - 8 &= 0 && \text{getting everything to one side} \\ u^2 + 2u - 8 &= 0 && \text{substituting } u = x^2 \\ (u+4)(u-2) &= 0 \\ u = -4 \text{ or } u = 2 \quad &\Rightarrow && x^2 = -4 \text{ or } x^2 = 2 \end{aligned}$$

But $x^2 = -4$ has no solutions, so $x = \pm\sqrt{2}$.

- For those who don't like substituting, just factorise and solve:

$$\begin{aligned} 2x^{\frac{2}{3}} &= 5x^{\frac{1}{3}} + 3 \\ 2x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 3 &= 0 \\ (2x^{\frac{1}{3}} + 1)(x^{\frac{1}{3}} - 3) &= 0 \end{aligned}$$

  So $x^{\frac{1}{3}} = -\frac{1}{2}$ or $x^{\frac{1}{3}} = 3$. Therefore cubing we find $x = -\frac{1}{8}$ or $x = 27$.

- Don't be one of the cretins who sees something like $x^4 + 4x^2 = 9$ and then *thinks* that they are square rooting to obtain $x^2 + 2x = 3$. Remember $\sqrt{x^4 + 4x^2} \neq x^2 + 2x$.
  Likewise $x + \sqrt{x} + 3 = 0$ does not square to $x^2 + x + 9 = 0$.

Differentiation

- We now turn to calculus[2]. Calculus ≡ Differentiation + Integration. You will discover integration in C2.
- Differentiation allows us to calculate the 'gradient function' $\frac{dy}{dx}$. This tells us how the gradient on the original function y changes with x. $\frac{dy}{dx}$ *is* the gradient of a curve. So if you need to find where on a curve the gradient is 7, then you solve $\frac{dy}{dx} = 7$.
- Two alternative notations for derivatives are $\frac{dy}{dx} \equiv f'(x) \equiv y'$.
- The "rules" are;

$$\begin{aligned} y &= \text{constant} && \Rightarrow && \frac{dy}{dx} = 0, \\ y &= ax && \Rightarrow && \frac{dy}{dx} = a, \\ y &= ax^n && \Rightarrow && \frac{dy}{dx} = anx^{n-1}. \end{aligned}$$

  Notice that the first two are merely subsets of the third; the third is the daddy; the big cheese; the head honcho…

- For example:

$$\begin{aligned} y = 4x^4 - 3x^2 + 2x - 5 \quad &\Rightarrow \quad \frac{dy}{dx} = 16x^3 - 6x + 2, \\ y = 4x^{\frac{5}{4}} + 3x^{\frac{4}{5}} \quad &\Rightarrow \quad \frac{dy}{dx} = 5x^{\frac{1}{4}} + \frac{12}{5}x^{-\frac{1}{5}}. \end{aligned}$$

[2] Isaac Newton. Arguably the greatest physicist ever. [Gottfried Leibniz also came up with it a bit later.]

- You must expand brackets or carry out divisions *before* you differentiate[3]. For example:

$$y = x^2(x-3)^2 \quad \Rightarrow \quad y = x^4 - 6x^3 + 9x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 18x,$$

$$y = \frac{x^7 + x}{x^6} \quad \Rightarrow \quad y = x + x^{-5} \quad \Rightarrow \quad \frac{dy}{dx} = 1 - 5x^{-6} = 1 - \frac{5}{x^6}.$$

- We can use differentiation to find the equation of tangents and normals to curves at specified points. For example find the equation of the normal to the curve $y = x^3 + 2x^2 - 5x - 1$ when $x = 1$.

  Firstly we need the y-coordinate: $x = 1 \quad \Rightarrow \quad y = -3$.

  Secondly $\frac{dy}{dx} = 3x^2 + 4x - 5$. Into this we put $x = 1$, so $\frac{dy}{dx} = 2$. Therefore the *normal* has gradient $-\frac{1}{2}$. So

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y + 3 &= -\tfrac{1}{2}(x-1) \\ x + 2y + 5 &= 0. \end{aligned}$$

- If asked to show that $5x + y + 17 = 0$ is tangent to the curve $y = x^2 + 3x - 1$, there are two methods to do this:

  1. Find where $y = x^2 + 3x - 1$ and $5x + y + 17 = 0$ cross. Solving simultaneously we gain the quadratic $-5x - 17 = x^2 + 3x - 1$ which simplifies and factorises to $(x+4)(x+4) = 0$. This gives a *repeated* root, so the line intersects the curve once and we can therefore conclude that the line *must* be a tangent. [I prefer this method.]
  2. The line $5x + y + 17 = 0$ has gradient -5. Therefore we need to find where on $y = x^2 + 3x - 1$ the gradient is -5. Therefore we differentiate $y = x^2 + 3x - 1$ to get $\frac{dy}{dx} = 2x + 3$ and put $\frac{dy}{dx} = -5$. This gives $x = -4$. On the curve, when $x = -4$, $y = 3$, so to find the equation of the tangent

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= -5(x+4) \\ x + 5y + 17 &= 0, \text{ as required.} \end{aligned}$$

- Another example: Given that the curve $y = ax^3 + 4x^2 + bx + 1$ passes through the point $(-1, 5)$ and (at that point) the tangent is parallel to the line $y + 4x + 1 = 0$. Find a and b.

  There's quite a bit going on here, so take it a bit at a time. Since the curve passes through $(-1, 5)$, then $x = -1$ and $y = 5$ must be a solution to the curve's equation, so $5 = -a + 4 - b + 1$ which simplifies to $a + b = 0$. The line given has gradient -4, so we need to set $\frac{dy}{dx} = -4$ when $x = -1$. So $\frac{dy}{dx} = 3ax^2 + 8x + b$ which gives $-4 = 3a - 8 + b$. These solve to $a = 2$, $b = -2$.

Inequalities

- Treat linear inequalities like equations except when multiplying or dividing by a negative number when you reverse the sign. For example

$$\begin{aligned} 2x + 4 &< 3x + 2 \\ -x &< -2 \\ x &> 2. \end{aligned}$$

[3] $y = \frac{x^3+x}{x^2}$ does *not* differentiate to $\frac{dy}{dx} = \frac{3x^2+1}{2x}$ and $y = (x^3+2)(2x^2+5x)$ does *not* differentiate to $\frac{dy}{dx} = 3x^2(4x+5)$!

- To solve quadratic inequalities:
  1. Get all terms over one side so that quadratic > 0 or quadratic < 0 in such a way that the x^2 term is always positive. This will ensure a 'happy' curve.
  2. Solve quadratic = 0 to find where it crosses x–axis.
  3. Sketch the graph and read off solution. If it is quadratic > 0 then it is the region(s) above the x–axis, and if quadratic < 0 then it is region below the x–axis.
  4. If one region then express as one triple inequality (e.g. $-2 < x < 5$) and if two regions then two *separate* inequalities (e.g. $x > 5$ or $x < -2$).
- For example solve the inequality $-7x \geqslant 4 - 2x^2$. Firstly get the $2x^2$ on the other side to make it positive to get $2x^2 - 7x - 4 \geqslant 0$. Then solve the equality $2x^2 - 7x - 4 = 0 = (2x+1)(x-4)$, so $x = -\frac{1}{2}$ or $x = 4$. So we have a happy quadratic that crosses the x–axis at $-\frac{1}{2}$ and 4. The inequality is asking for where the curve is bigger than (or equal to) zero, and this is to the right of $x = 4$ and the left of $x = -\frac{1}{2}$. Therefore the solution is $x \leqslant -\frac{1}{2}$ or $x \geqslant 4$.
- Don't fall into the trap of seeing $x^2 < 16$ and saying $x < \pm 4$! Be disciplined and get zero on one side; $x^2 - 16 < 0$ so $(x-4)(x+4) < 0$ so we have happy curve that crosses at 4 and −4. Where is the curve less than zero? Between −4 and 4 so solution is $-4 < x < 4$.

Index Notation

- $(ab)^m = a^m \times b^m$. For example $6^5 = 2^5 \times 3^5$.
- When multiplying a number raised to different powers the powers *add*. Therefore $a^m \times a^n = a^{m+n}$. You can think of this as follows $2^2 \times 2^4 = (2 \times 2) \times (2 \times 2 \times 2 \times 2) = 2^6$.
- Know that $a^{-m} = \frac{1}{a^m}$. Remember this by the standard result that $2^{-1} = \frac{1}{2}$. "When moving something from the bottom line of a fraction to the top (or vice versa), the sign changes."
- From the above two results we can obtain the result $\frac{a^m}{a^n} = a^{m-n}$. This is derived thus; $\frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}$ as required.
- We can also derive the important result $a^0 = 1$ for any $a \neq 0$. Derived by considering something like this; $a^0 = a^{1-1} = \frac{a^1}{a^1} = \frac{a}{a} = 1$.
- Know that $(a^m)^n = a^{mn}$. Think about it like this; $(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{12}$.
- The n^{th} root of a number can be expressed as a power thus; $\sqrt[n]{a} = a^{\frac{1}{n}}$.
- A few examples:
  1. Write 8 as a power of 4; well $8 = 2^3 = \left(4^{\frac{1}{2}}\right)^3 = 4^{\frac{3}{2}}$.
  2. Simplify $\sqrt[4]{16^3} = \left(\left(2^4\right)^3\right)^{\frac{1}{4}} = \left(2^{12}\right)^{\frac{1}{4}} = 2^3 = 8$.
  3. Simplify $\dfrac{12x^8y^{\frac{3}{2}}}{6x^6y^{\frac{5}{2}}} = 2x^{8-6}y^{\frac{3}{2}-\frac{5}{2}} = 2x^2y^{-1}$.
  4. Simplify $\sqrt{x^6y^4} \times \sqrt[3]{x^3y^{-6}} = \left(x^6y^4\right)^{\frac{1}{2}} \times \left(x^3y^{-6}\right)^{\frac{1}{3}} = x^3y^2x^1y^{-2} = x^4$.
- OCR is particularly 'hot' on linking differentiation, indices and surds. For example, find the equation of the normal to $y = 6x^{\frac{5}{2}} - 4x^{\frac{3}{2}}$ when $x = 2$ in the form $ax + by = c$.

  When $x = 2$, $y = 6 \times 2^{\frac{5}{2}} - 4 \times 2^{\frac{3}{2}} = 24\sqrt{2} - 8\sqrt{2} = 16\sqrt{2}$.

Differentiating we find $\frac{dy}{dx} = 15x^{\frac{3}{2}} - 6x^{\frac{1}{2}}$.

So, when $x = 2$, $\frac{dy}{dx} = 15 \times 2\sqrt{2} - 6\sqrt{2} = 24\sqrt{2}$. Therefore the gradient of the normal is $-\frac{1}{24\sqrt{2}} = -\frac{\sqrt{2}}{48}$. Therefore the equation of the normal is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 16\sqrt{2} &= -\tfrac{\sqrt{2}}{48}(x - 2) \\ \sqrt{2}x + 48y &= 770\sqrt{2}. \end{aligned}$$

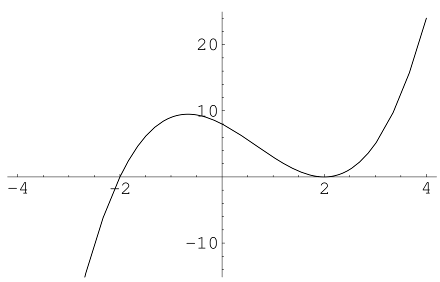
Graphs of *n*th Power Functions

- Be able to sketch $y = \frac{1}{x^n}$ for $n = \{1, 2, 3 \ldots\}$.
  If n is even then graphs look like $y = \frac{1}{x^2}$ with the y-axis being a line of symmetry.
  If n is odd then the graph looks like $y = \frac{1}{x}$.
- Shock, horror! The differentiation rule for $y = ax^n$ still works with fractional and negative powers:

$$\frac{dy}{dx} = anx^{n-1}.$$

Polynomials

- Must be able to visualise a polynomial curve quickly. This is determined by two things;
  1. Whether the largest power of x is odd or even. x^{1000} has very different shape from x^{1001}.
  2. Whether the coefficient (number in front of) the largest power of x is positive or negative.
- A polynomial of order n has *at most* $n - 1$ stationary points. For example a cubic curve has up to two stationary points, but it might have none or one.
- If $y = x^3 + 2x^2 + 3x + 4$ then the curve crosses the y-axis at 4 (just the constant at the end). To find the x-axis intercept(s) need to solve $0 = x^3 + 2x^2 + 3x + 4$.
- To sketch a polynomial it is best to factorise it. For example given $y = x^3 - 2x^2 - 4x + 8$ we can write $y = (x + 2)(x - 2)^2$. So to sketch we know that it is a cubic with positive x^3 coefficient. y-axis intercept is at 8. It crosses the x-axis at $x = -2$, but only touches it at $x = 2$ due to the repeated root.



Transforming Graphs

- Given $y = f(x)$ then:

Replacement	Graph Shape
None	Normal Graph
x by $x - a$	Graph translated by a vector $\binom{a}{0}$
x by $-x$	Graph reflected in the y–axis
x by $\frac{x}{2}$	Graph stretched by a factor of 2 parallel to the x–axis ←→
x by $2x$	Graph stretched by a factor of $\frac{1}{2}$ parallel to the x–axis →←
y by $y - b$	Graph translated by a vector $\binom{0}{b}$
y by $-y$	Graph reflected in the x–axis
y by $\frac{y}{2}$	Graph stretched by a factor of 2 parallel to the y–axis ↕
y by $2y$	Graph stretched by a factor of $\frac{1}{2}$ parallel to the y–axis ↓↑

- For example: Find the equation of $y^2 + 2x^2 = 2x + 1$ after the translation $\binom{1}{-1}$. So we replace x by $x - 1$ and y by $y + 1$. Therefore

$$(y+1)^2 + 2(x-1)^2 = 2(x-1) + 1 \quad \Rightarrow \quad y^2 + 2y + 2x^2 = 6x - 4.$$

- For example: Explain the transformation that maps

$$y = \frac{2}{\sqrt{1+x}} \text{ onto } y = \frac{1}{\sqrt{3+x}}.$$

Rewriting the second equation as $2y = \frac{2}{\sqrt{1+(x+2)}}$ we can see y has been replaced by $2y$ and x has been replaced by $x+2$. Therefore the curve has been translated by $\binom{-2}{0}$ and also stretched by a factor of $\frac{1}{2}$ parallel to the y–axis; i.e. every y–value has halved.

Investigating Shapes of Graphs

- Stationary points are where the gradient of curve is zero. They are either maxima, minima or points of inflection. To find the turning points of a curve we must find $\frac{dy}{dx}$ and then set $\frac{dy}{dx} = 0$ and solve for x.

- To determine the nature of a turning point we can consider the sign of the gradient either side of the turning point. Present this in a table. In the example of $y = x^2 + 2x + 3$ we find $\frac{dy}{dx} = 2x + 2$ so we solve $0 = 2x + 2$ to give the turning point when $x = -1$:

x	$x < -1$	-1	$x > -1$
dy/dx	negative	0	positive
		minimum	

- We can also use the second derivative to determine the nature of a turning point. This is found by differentiating the function twice;

$$y = 2x^3 + 3x^2 - 2x + 4 \quad \Rightarrow \quad \frac{dy}{dx} = 6x^2 + 6x - 2 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 12x + 6.$$

You then evaluate the second derivative with the x value at the turning point and look at its sign. If it is positive it is a minimum, if it is negative it is a maximum. If it is zero then it is *probably* a point of inflection, but you need to do the above analysis either side of the turning point.

- For example, determine the nature of the stationary points on $y = 4x^3 - 21x^2 + 18x + 3$. So

$$y = 4x^3 - 21x^2 + 18x + 3 \quad \Rightarrow \quad \frac{dy}{dx} = 12x^2 - 42x + 18 = 0 \quad \Rightarrow \quad x = 3 \text{ or } x = \tfrac{1}{2}.$$

  Therefore the stationary points are $(3, -24)$ and $(\frac{1}{2}, \frac{29}{4})$. We therefore need the second derivative and evaluate it at 3 and $\frac{1}{2}$.

$$\frac{d^2y}{dx^2} = 24x - 42$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=3} = 24 \times 3 - 42 = 30 > 0 \text{ therefore } (3, -24) \text{ is a minimum.}$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = 24 \times \tfrac{1}{2} - 42 = -30 < 0 \text{ therefore } (\tfrac{1}{2}, \tfrac{29}{4}) \text{ is a maximum.}$$

- A function will be *increasing* when $\frac{dy}{dx}$ is positive and *decreasing* when $\frac{dy}{dx}$ is negative[4]. This is obvious if you consider a sketch.

  For example, find the set of values of x for which $y = -2x^3 + 3x^2 + 12x + 1$ is decreasing. First differentiate and recognise we want $\frac{dy}{dx}$ to be negative.

$$y = -2x^3 + 3x^2 + 12x + 1$$
$$\frac{dy}{dx} = -6x^2 + 6x + 12$$
$$0 > -6x^2 + 6x + 12 \qquad \text{(note that we place } \frac{dy}{dx} < 0\text{)}$$
$$x^2 - x - 2 > 0$$
$$(x - 2)(x + 1) > 0.$$

  This quadratic inequality solves to $x < -1$ or $x > 2$ which are the values of x for which the curve is decreasing.

Applications of Differentiation

- Differentiation can be used to work out "rates of change". In GCSE Physics you learnt that acceleration is the rate of change of velocity; you will then have learnt that $a = \frac{v-u}{t}$. However, at a higher level rates of change are calculated by differentiating with respect to time. So we now view acceleration as $a = \frac{dv}{dt}$.

- As we have already seen, differentiation allows us to calculate the stationary point of a curve $y = f(x)$. We do this by calculating $\frac{dy}{dx}$ and setting it equal to zero. We can use this to help is in practical problems where we might want to maximise a quantity (e.g. profit) or to minimise a quantity (e.g. cost).

- Worked example: An open topped cuboidal box is to be made from a rectangular piece of metal 10cm by 16cm. Squares are to be cut from each corner and then the four flaps are to be folded up. Find the maximum volume attainable for the box and prove that it is a maximum.

  1. Let x be the side length of the squares cut away, where $0 < x < 5$.

[4]Technically a curve is 'increasing' when $\frac{dy}{dx} \geqslant 0$ and 'strictly increasing' when $\frac{dy}{dx} > 0$; likewise for decreasing, but don't get too het up about it; just get the sign the right way round.

2. The volume of the box would therefore be

$$V = x(16 - 2x)(10 - 2x) = 160x - 52x^2 + 4x^3.$$

We imagine a graph of V against x and hope that there is a stationary point in the range $0 < x < 5$.

3. Differentiate V with respect to x and set equal to zero to find the stationary point;

$$\frac{dV}{dx} = 160 - 104x + 12x^2 = 0 \quad \Rightarrow \quad x = 2 \text{ or } x = \tfrac{20}{3}.$$

4. Notice that x can't be $\frac{20}{3}$ because it is outside the range $0 < x < 5$. So we only consider $x = 2$.
5. If $x = 2$ then $V = 2 \times 12 \times 6 = 144$.
6. To demonstrate that $x = 2$ is a maximum we need the second derivative and evaluate it at $x = 2$.

$$\frac{d^2V}{dx^2} = -104 + 24x = -104 + 24 \times 2 = -56 < 0 \text{ therefore a maximum.}$$

Circles

- Circles with centre $(0, 0)$ and radius r are expressed by $x^2 + y^2 = r^2$.
- Circles with centre (a, b) and radius r are expressed by $(x - a)^2 + (y - b)^2 = r^2$.
- By 'completing the square' you can convert circles of the form $x^2 + y^2 + \alpha x + \beta y + \gamma = 0$ into the form $(x - a)^2 + (y - b)^2 = r^2$. For example

$$\begin{aligned} x^2 + y^2 + 6x - 4y + 9 &= 0 \\ x^2 + 6x + y^2 - 4y + 9 &= 0 \\ (x + 3)^2 - 9 + (y - 2)^2 - 4 + 9 &= 0 \\ (x + 3)^2 + (y - 2)^2 &= 4. \end{aligned}$$

- When finding the intersection of a line and a circle it is easiest to substitute in the value of y from the line into the circle and solve the resulting quadratic. For example; find where the line $y = 2x - 1$ intersects to circle $(x - 3)^2 + (y - 2)^2 = 25$.

$$\begin{aligned} (x - 3)^2 + (y - 2)^2 &= 25 \\ (x - 3)^2 + (2x - 3)^2 &= 25 \\ x^2 - 6x + 9 + 4x^2 - 12x + 9 - 25 &= 0 \\ 5x^2 - 18x - 7 &= 0. \end{aligned}$$

  Solve the quadratic (in this case by the formula) and then find the y values by substituting both x values into $y = 2x-1$ (the original line). There will usually be 2 points of intersection (where the discriminant of the resulting quadratic will be positive) except if the line doesn't intersect the circle at all (discriminant negative) or if the line is a tangent to the circle (discriminant equals zero).

- The gradient of the tangent to a circle is perpendicular to the radius of the circle at that point. For example: The point $B(1, 7)$ lies on the circle $(x - 3)^2 + (y - 4)^2 = 13$. Find the equation of the tangent to the circle at B. The centre of the circle is $(3, 4)$, so the gradient of the radius at B is $-\frac{3}{2}$. Therefore the gradient of the tangent is $\frac{2}{3}$ and will pass through B, so the tangent will be:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 7 &= \tfrac{2}{3}(x - 1) \\ 0 &= 2x - 3y + 19. \end{aligned}$$

- You must always remember the GCSE theorem that if a triangle is constructed within a circle with one side being a diameter of the circle, then it is a right angled triangle. To demonstrate this one is often required to show that the gradients of certain line segments are perpendicular (i.e. $m_1 \times m_2 = -1$).

  For example; $A(2, 1)$, $B(4, 13)$ and $C(-3, 8)$. The line segment AB is the diameter of a circle and C is a point on its circumference. Find the area of triangle ABC. We know angle $A\hat{C}B$ must be a right angle, so

$$\begin{aligned} \text{Area } ABC &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times (\text{length } AC) \times (\text{length } CB) \\ &= \tfrac{1}{2} \times \sqrt{5^2 + 7^2} \times \sqrt{7^2 + 5^2} \\ &= 37 \text{ units}^2. \end{aligned}$$

- Also, given two points that lie on a circle's circumference, the centre of the circle lies on the perpendicular bisector of the two points.

Trigonometry

- We define

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}.$$

  This identity is very useful in solving equations like $\sin\theta - 2\cos\theta = 0$ which yields $\tan\theta = 2$. The solutions of this in the range $0° \leqslant \theta \leqslant 360°$ are $\underline{\theta = 63.4°}$ and $\underline{\theta = 243.4°}$ to one decimal place.

- Know the following (or better yet, learn a couple and be able to derive the rest, quickly, from your knowledge of the trigonometric functions):

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	1	0
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	1	0	undefined
$180°$	0	-1	0

- Be able to sketch $\sin\theta$, $\cos\theta$ and $\tan\theta$ in both degrees and radians.

- By considering a right angled triangle (or a point on the unit circle) we can derive the important result $\sin^2\theta + \cos^2\theta \equiv 1$. This is useful in solving certain trigonometric equations.

Worked example; solve $1 = 2\cos^2\theta + \sin\theta$ for $0° \leqslant \theta \leqslant 360°$.

$$\begin{aligned} 1 &= 2\cos^2\theta + \sin\theta \\ 1 &= 2(1-\sin^2\theta) + \sin\theta && \text{get rid of } \cos^2\theta, \\ 0 &= 1 - 2\sin^2\theta + \sin\theta && \text{quadratic in } \sin\theta, \\ 0 &= 2\sin^2\theta - \sin\theta - 1 && \text{factorise as normal,} \\ 0 &= (2\sin\theta + 1)(\sin\theta - 1). \end{aligned}$$

So we just solve $\sin\theta = -\frac{1}{2}$ and $\sin\theta = 1$. Therefore $\underline{\theta = 210°}$ or $\underline{\theta = 330°}$ or $\underline{\theta = 90°}$.

- The above relation is also useful in converting between the different trigonometric functions. For example if $\cos\theta = \frac{6}{7}$ then, to find $\sin\theta$, do **not** use "$\cos^{-1}$" on your calculator and then "sin" the answer. Instead

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1, \\ \sin^2\theta + \frac{36}{49} &= 1, \\ \sin\theta &= \pm\sqrt{\frac{13}{49}} = \pm\frac{\sqrt{13}}{7}. \end{aligned}$$

  Without further information you must keep both the positive and negative solution.

- If a question tells you that the angle is 'acute', 'obtuse' or 'reflex' then you must visualise the appropriate graph and interpret. For example given that $\sin\theta = \frac{1}{3}$ and that θ is obtuse, find the value of $\cos\theta$. By the argument above you will find that

$$\cos\theta = \pm\frac{\sqrt{8}}{3} = \pm\frac{2\sqrt{2}}{3}.$$

  However, given an obtuse angle ($90° < \theta < 180°$) the cosine graph is negative, so the final answer should be $\cos\theta = -\frac{2\sqrt{2}}{3}$.

- You must be careful when you see things like $2\tan x \sin x = \tan x$. It is SO tempting to divide both sides by $\tan x$ to yield $2\sin x = 1$. But you must bring everything to one side and factorise;

$$2\tan x\sin x - \tan x = 0 \qquad \Rightarrow \qquad \tan x(2\sin x - 1) = 0.$$

  The full set of solutions can then be by solving $\tan x = 0$ and $2\sin x - 1 = 0$. [It is completely analogous to $x^2 = x$. If we divide by x we find $x = 1$, but we know this has missed the solution $x = 0$. However when we factorise we find $x(x-1) = 0$ and both solutions are found.]

- Given a trigonometric equation it is always best first to isolate the trigonometric function on its own; for example

$$9\cos(\ldots) + 2 = 7 \qquad \Rightarrow \qquad \cos(\ldots) = \tfrac{5}{9}.$$

- For complicated trigonometric equations where you are not just 'cos'ing, 'sin'ing or 'tan'ing a single variable (x, θ, t or the like), it is often easiest to make a substitution.

  For example to solve $\cos(2x + 30) = \frac{1}{4}$ in the range $0° \leqslant x \leqslant 360°$ the desired substitution is clearly $u = 2x + 30$, but you **must** remember to also convert the range also (many students forget this) so:

$$\begin{aligned} \cos(2x+30) &= \tfrac{1}{4} && 0° \leqslant x \leqslant 360°, \\ \cos u &= \tfrac{1}{4} && 30° \leqslant u \leqslant 750°, \\ u &= \ldots°, \ldots°, \ldots°, \ldots°. \end{aligned}$$

However, we don't want solutions in u, so we need to use $x = \frac{u-30}{2}$ on each u solution to get

$$x = \ldots^\circ, \ldots^\circ, \ldots^\circ, \ldots^\circ.$$

Sequences

- You must be comfortable with $\sum$-notation. It works as follows; you put in the number at the bottom of the $\sum$ and then keep summing until you reach the top number[5]. For example:

$$\sum_{i=4}^{8}(2i+3) = 11 + 13 + 15 + 17 + 19 = 75,$$
$$\sum_{i=1}^{n}(i^2+i) = (1+1) + (4+2) + (9+3) + \cdots + (n^2+n).$$

- A 'sequence' is a list of numbers in a specific order. A 'series' is a sum of the terms of a sequence.
- Sequences are sometimes defined *recursively*. For example the sequence $a_{n+1} = a_n + 3$ with $a_1 = 10$ defines the sequence $10, 13, 16, 19\ldots$ We know that this is an arithmetic sequence which can also be defined *deductively* by $a_n = 10 + 3(n-1)$.
- An arithmetic sequence increases or decreases by a constant amount. The letter a always denotes the first term and d is the difference between the terms (negative for a decreasing sequence!). The nth term is denoted a_n and satisfies the important relationship

$$a_n = a + (n-1)d.$$

  For example if told the third term of a sequence is 10 and the seventh term is 34 then we can use the above equation to find the a and d.

$$\begin{matrix} 10 = a + (3-1)d \\ 34 = a + (7-1)d \end{matrix} \quad \Rightarrow \quad 4d = 24 \quad \Rightarrow \quad d = 6 \quad \Rightarrow \quad a = -2.$$

- The sum of the n terms of an arithmetic sequence is given by

$$S = \frac{n}{2}(\text{First} + \text{Last}) = \frac{n}{2}(2a + (n-1)d).$$

  For example the sum of the first 10 terms of a sequence is 130 and the first term is 4. What is the difference?

$$S = \frac{n}{2}(2a + (n-1)d) \quad \Rightarrow \quad 130 = \frac{10}{2}(8 + (10-1)d) \quad \Rightarrow \quad d = 2.$$

Binomial Theorem

- Binomial expansion allows us to expand $(a+b)^n$ for any integer n. Best explained by means of an example; expand $(2x-y)^5$.

  1. Begin by considering 'prototype' expansion of $(a+b)^5$.
  2. So $(a+b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5$.

---

[5] If you want to multiply instead then use $\prod$-notation. For example $\prod_{i=1}^{n} i = 1 \times 2 \times 3 \times 4 \times \cdots \times n \equiv n!$

3. Calculate binomial coefficients either on calculator or by drawing a mini Pascal's Triangle to give $(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$.
4. Next notice that in our case $a = 2x$ and $b = -y$ and substitute in to get $(2x-y)^5 = 1(2x)^5 + 5(2x)^4(-y) + 10(2x)^3(-y)^2 + 10(2x)^2(-y)^3 + 5(2x)(-y)^4 + 1(-y)^5$.
5. Tidying up we get $(2x-y)^5 = 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$.

- It is worth noting that when the expansion is of the form (something – another thing)^n, then the signs will alternate.
- Also of note is the way each *individual* component is constructed. For example; find the x^5 coefficient in the expansion of $(2-3x)^7$. The component with x^5 is given by $\binom{7}{5}(2)^2(-3x)^5 = -20412x^5$, so the coefficient is -20412.
- $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$. For example $\binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5\times4\times3\times2\times1}{(2\times1)\times(3\times2\times1)} = 10$.
- Know that $(1+x)^n$ expands thus:

$$\begin{aligned}(1+x)^n &= 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + x^n \\ &= 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n.\end{aligned}$$

  (This is particularly useful in Core 4.)

Sine & Cosine Rules

- The sine rule states for any triangle $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
- The cosine rule states that $a^2 = b^2 + c^2 - 2bc\cos A$. Practice both sine and cosine rules on page 293.
- By considering half of a general parallelogram we can show that the area of any triangle is given by $A = \frac{1}{2}ab\sin C$.
- You must be good at bearing problems which result in triangles. Remember to draw lots of North lines and remember also that they are all parallel; therefore you can use Corresponding, Alternate and Allied angle theorems…revise your GCSE notes! Bearings are measured clockwise from North and must contain three digits. For example

$$12.2^\circ \quad\Rightarrow\quad 012.2^\circ.$$

Integration

- *Calculus* is the combined study of differentiation *and* integration (and their relationship). A good description is that calculus is the study of change in the same way that geometry is the study of shapes.
- Integration is the reverse of differentiation. That is if $\frac{dy}{dx} = f(x)$ then $y = \int f(x)\,dx$. For example if $\frac{dy}{dx} = 3x^3$ then $y = \int 3x^3\,dx = \frac{3}{4}x^4 + c$.
- The general rule is therefore $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$.

- $\int y\,dx$ is an *indefinite* integral because there are no limits on the integral sign. When evaluating these integrals *never* forget an *arbitrary constant* added on at the end. For example $\int 6x^2\,dx = 2x^3 + c$.

- $\int_a^b y\,dx$ is a *definite integral* and is the area between the curve and the x-axis from $x = a$ to $x = b$. Areas under the x-axis are negative. (For areas between the curve and the y-axis switch the x and the y and use $\int_p^q x\,dy$ between $y = p$ and $y = q$.)

- To find the area *between* two curves between $x = a$ and $x = b$ evaluate

$$\int_a^b (\text{top} - \text{bottom})\,dx.$$

- For example, given that $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, find the area under the curve from $x = 1$ and $x = 2$.

$$\begin{aligned}\int_1^2 \sqrt{x} + \frac{1}{\sqrt{x}}\,dx &= \int_1^2 x^{\frac{1}{2}} + x^{-\frac{1}{2}}\,dx\\ &= \left[\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right]_1^2\\ &= \left(\frac{2}{3}\times 2^{\frac{3}{2}} + 2\times 2^{\frac{1}{2}}\right) - \left(\frac{2}{3} + 2\right)\\ &= \frac{10\sqrt{2}}{3} - \frac{8}{3}.\end{aligned}$$

- To calculate integrals where one of the limits is infinite (∞ or $-\infty$), proceed as normal until you input the ∞ or $-\infty$ into the integral. Then you must **not** write such things as

$$\frac{1}{\infty},\quad \frac{1}{3^\infty},\quad \frac{2\infty + 1}{3\infty - 2},\quad 2^{\frac{1}{\infty}},\quad \text{and the like.}$$

  You must think what these things would equal and just write down the number; in the previous four cases you would get 0, 0, $\frac{2}{3}$ and 1. (If, in the C2 exam, you think that when you put in ∞ you get an infinite answer, chances are you've made a mistake somewhere.) For example:

$$\begin{aligned}\int_2^\infty \frac{8}{5x^3}dx &= \int_2^\infty \frac{8}{5}x^{-3}dx\\ &= \left[-\frac{4}{5}x^{-2}\right]_2^\infty\\ &= (0) - \left(-\frac{4}{5}\times\frac{1}{2^2}\right) = \frac{1}{5}.\end{aligned}$$

Geometric Sequences

- A geometric sequence is one where the terms are multiplied by a constant amount. For example $1, 2, 4, 8, 16, \ldots, [2^{n-1}]$ is a geometric sequence with $a = 1$ and $r = 2$. The n^{th} term is given by

$$a_n = ar^{n-1}.$$

  So for the above example the 20th term is $a_{20} = 1 \times 2^{19} = 524288$.

- The sum of n terms of a geometric sequence is given by

$$S = a\left(\frac{r^n - 1}{r - 1}\right) \qquad \text{or (equivalently) by} \qquad S = a\left(\frac{1 - r^n}{1 - r}\right).$$

  For example sum the first 20 terms of $4, 2, 1, \frac{1}{2}, \ldots, [4 \times \left(\frac{1}{2}\right)^{n-1}]$. This is given by

$$S = 4\left(\frac{\left(\frac{1}{2}\right)^{20} - 1}{\frac{1}{2} - 1}\right) = 7.999992371\ldots$$

- If the ratio (r) lies between -1 and 1 (i.e. $-1 < r < 1$) then there exists a 'sum to infinity' given by

$$S_\infty = \frac{a}{1 - r}.$$

  Therefore S_∞ for the above example is $S_\infty = \frac{4}{1-\frac{1}{2}} = 8$. We can see that the sum to 20 terms is very close to S_∞.

Exponentials & Logarithms

- (In these notes if I write $\log x$ I *mean* $\log_{10} x$. If I mean a different base, I will write it *explicitly* as $\log_a x$. When we see $\log_a x$ we say "log to the base a of x".)
- We *define* a logarithm to be the solution (in x) to the equation $a^x = b$. It is written $x = \log_a b$. The fundamental relationship is therefore

$$a^x = b \qquad \Leftrightarrow \qquad x = \log_a b. \qquad\qquad \dagger$$

- From † we see that $\log_a b$ *means*: "The number a has to be raised to, to make b". Therefore some simple logarithms can be calculated without a calculator:

$$\begin{aligned} \log_2 8 &= \text{"the number 2 has to be raised to, to make 8"} &&= 3, \\ \log_{10} 10000 &= \text{"the number 10 has to be raised to, to make 10000"} &&= 4, \\ \log_9 3 &= \text{"the number 9 has to be raised to, to make 3"} &&= \tfrac{1}{2}, \quad (\because 3 = \sqrt{9} = 9^{\frac{1}{2}}) \\ \log_a a &= \text{"the number } a \text{ has to be raised to, to make } a\text{"} &&= 1. \end{aligned}$$

- We see from † that logarithms 'pluck out powers' from equations. Therefore if you ever see an equation with the unknown in the power, then that is the clue that you will need to use logarithms. For example to solve $7^{2x-1} = 22$ we discover

$$\begin{aligned} 7^{2x-1} &= 22, \\ 2x - 1 &= \log_7 22, \\ x &= \tfrac{1}{2}\log_7 22 + \tfrac{1}{2}. \end{aligned}$$

  However we need to build on † because not all equations are this simple (e.g. $3 \times 2^{2x-1} = 5 \times 7^{x+1}$) and not all calculators can calculate $\log_7 22$.

- You can also use † to eliminate logarithms from an equation. Given an equation of the form $\log_a(\cdots) = b$, you can eliminate the logarithm instantly to get $(\cdots) = a^b$. A good way to remember this[6] is 'Girvan's Bullying Base'. So if we have $\log_3 x = 8$, then the bullying base '3' knocks the log out of the way and moves to the other side and squeezes up the 8 to put it in its place; therefore $x = 3^8$.

[6]From a colleague I respect; Mr Girvan

- Logarithms and exponentials (powers) are the inverse functions of each other (as can be seen from † if one puts one into the other). Therefore

$$\log_{10} 10^x = x \quad \text{and} \quad 10^{\log_{10} x} = x.$$

So if $\log a = 5.4$ then $a = 10^{5.4}$.

- There are some rules that can be derived from † that *must* be learnt. They are (for all bases):

$$\log(ab) = \log a + \log b \qquad \log 1 = 0$$
$$\log\left(\frac{a}{b}\right) = \log a - \log b \qquad \log_a a = 1$$
$$\log(a^n) = n \log a \qquad \log_a b = \frac{\log_c b}{\log_c a}.$$
$$\log\left(\frac{1}{a}\right) = -\log a$$

- When we need to solve an equation where the unknown is in the exponent such as $5^{2x-1} = 8$ take $\log_{10}$ of both sides and simplify:

$$\begin{aligned} 5^{2x-1} &= 8 \\ \log_{10}(5^{2x-1}) &= \log_{10} 8 \\ (2x-1)\log_{10} 5 &= \log_{10} 8 \\ 2x - 1 &= \frac{\log_{10} 8}{\log_{10} 5} \\ x &= \frac{1}{2} \times \left(\frac{\log_{10} 8}{\log_{10} 5} + 1\right) \\ x &= 1.15 \text{ (3sf).} \end{aligned}$$

Factors and Remainders

- Need to know how to divide any polynomial by a linear factor of the form $ax - b$. For example divide $x^3 + 2x^2 + 3x - 6$ by $x - 2$. (*Always* devote a column to each power of x.)

$$\begin{array}{r|rrrr} & & x^2 & +4x & +11 \\ \hline x-2 & +x^3 & +2x^2 & +3x & -6 \\ & +x^3 & -2x^2 & & \\ \hline & & +4x^2 & +3x & \\ & & +4x^2 & -8x & \\ \hline & & & +11x & -6 \\ & & & +11x & -22 \\ \hline & & & & +16 \end{array}$$

So the remainder is +16. Therefore $x^3 + 2x^2 + 3x - 6 = (x-2)(x^2 + 4x + 11) + 16$.

- If the remainder is zero, then the divisor is said to be a *factor* of the original polynomial.
- The Factor Theorem states:

$$(x - a) \text{ is a factor of } f(x) \qquad \Leftrightarrow \qquad f(a) = 0.$$

[More generally (but used less often in exams) is:

$$(ax - b) \text{ is a factor of } f(x) \qquad \Leftrightarrow \qquad f(\tfrac{b}{a}) = 0.]$$

- For example if $x^3 + ax^2 + 8x - 4$ has $(x - 2)$ as a factor, find a. From factor theorem we know $f(2) = 0$, so we discover $2^3 + a \times 2^2 + 8 \times 2 - 4 = 0$, and therefore $a = -5$.
- The Remainder Theorem states:

  When $f(x)$ is divided by $(x - a)$ the remainder is $f(a)$.

  [More generally (but used less often in exams) is:

  When $f(x)$ is divided by $(ax - b)$ the remainder is $f(\frac{b}{a})$.]

  Notice that the factor theorem is a subset of the remainder theorem. In the factor theorem all remainders are zero, by definition.
- For example if told that when $f(x) = x^3 + 2x^2 - 3x - 7$ is divided by $x - 2$ the remainder is 3, we know $f(2) = 3$.
- Worked example: $f(x) = 2x^3 + 3x^2 + kx - 2$. The remainder when $f(x)$ is divided by $(x - 2)$ is four times the remainder when $f(x)$ is divided by $(x + 1)$. Find k. We know

$$\begin{aligned} f(2) &= 4 \times f(-1) \\ 2 \times 2^3 + 3 \times 2^2 + 2k - 2 &= 4[2 \times (-1)^3 + 3 \times (-1)^2 - k - 2] \\ k &= -5. \end{aligned}$$

Radians

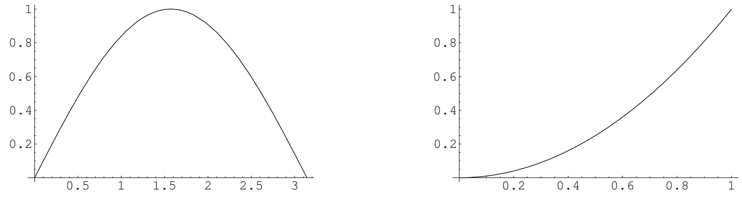
- There are (by definition) 2π radians in a circle. So $360° = 2\pi$. To convert from degrees to radians we use the conversion factor of $\frac{\pi}{180}$. For example to convert 45° to radians we calculate $45 \times \frac{\pi}{180} = \frac{\pi}{4}$ rad. From radians to degrees we use its reciprocal $\frac{180}{\pi}$.
- *When using radians* the formulae for arc length and area of a sector of a circle become simpler. They are $s = r\theta$ and $A = \frac{1}{2}r^2\theta$.

The Trapezium Rule

- The area under *any* curve can be *approximated* by the Trapezium Rule. The governing formula is given by (and contained in the formula booklet you will have in the exam)

$$\int_a^b y\, dx \approx \frac{1}{2}h\left[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\right],$$

  where h is the width of each trapezium, y_0 and y_n are the 'end' heights and $y_1 + y_2 + \cdots + y_{n-1}$ are the 'internal' heights.
- By considering the shape of the graph in the interval over which you are approximating it should be clear whether your estimate of the area is an over or under-estimate of the *true* area.



  For example if you were to estimate $\int_0^\pi \sin x\, dx$ (above, left) using the trapezium rule, due to the shape of the curve, the trapezia would all fall below the curve, so we would obtain an *under*-estimate. However, with $\int_0^1 x^2\, dx$ (above, right) we would obtain an *over*-estimate.

Functions

- A *function* is a *one-to-one* or a *many-to-one mapping*. There are also *many-to-many* and *one-to-many* mappings, but these are ***not*** functions. In a function, for every value you feed into the function you obtain one (and only one) value out.

- The *domain* of a function $y = f(x)$ is all the possible values of x the function can take. For example the domain of $y = \sqrt{x-4}$ is $x \geqslant 4$. In other words all the *inputs* the function can take.

- The *range* of a function is all the possible *outputs*. That is all the possible values of $f(x)$. So for $f(x) = -x^2 + 5$ the range is $f(x) \leqslant 5$.

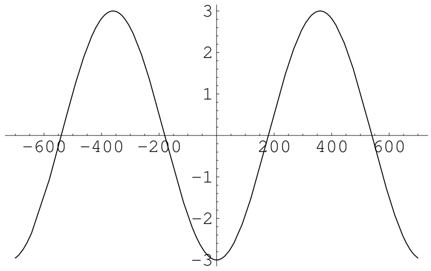
- Functions are transformed as follows

FUNCTION	GRAPH SHAPE
$f(x)$	Normal Graph
$2f(x)$	Graph stretched by a factor of 2 parallel to the y-axis i.e. every value of $f(x)$ in the original graph is multiplied by 2
$f(2x)$	Graph stretched by factor of $\frac{1}{2}$ parallel to the x-axis
$3f(\frac{x}{4})$	Graph stretched by factor of 4 parallel to the x-axis and a stretch by a factor of 3 parallel to the y-axis
$f(x) + 6$	Graph translated vertically *up* 6 units
$f(x) - 6$	Graph translated vertically *down* 6 units
$f(x + 4)$	Graph translated 4 units to the *left*
$f(x - 6)$	Graph translated 6 units to the *right*
$f(x - 6) + 9$	Graph translated 6 units to the *right* and 9 units *up*. This is a translation and can be expressed as $\binom{6}{9}$ where $\binom{\text{change in } x}{\text{change in } y}$
$-f(x)$	Graph reflected in the x-axis
$f(-x)$	Graph reflected in the y-axis

- When faced with more than one of the above transformations it sometimes matters which order you carry out the transformations. In the example of $2f(x-3)$ it doesn't matter because you end up with the same result both ways, regardless of whether you do the translation right, or the stretch parallel to the y-axis first (think about it). However with $f(2x + 10)$ you get a different result depending on the order you carry out the translation 10 left and then stretch factor $\frac{1}{2}$ parallel to the x-axis. If the conflict occurs within the bracket you should do the *opposite* of what you expect. So here you do the translation first and then the stretch.

  For $2f(x) + 6$ the transformations are outside the bracket, so here you would do the stretch *then* the translation.

- So for example if you were asked to sketch $y = 3\sin(\frac{x}{2} - 90)$ you would translate '$y = \sin x$' 90° to the right, *then* stretch factor 2 parallel to the x-axis and stretch factor 3 parallel to the y-axis.



- If $f(x) = f(-x)$ then the function is called an *even* function. An even function is one where the y-axis is a line of symmetry. Examples are

$$\begin{aligned} f(x) &= \cos x && \text{since} && f(-x) = \cos(-x) = \cos x = f(x), \\ g(x) &= x^2 + 1 && \text{since} && g(-x) = (-x)^2 + 1 = x^2 + 1 = g(x). \end{aligned}$$

- If $-f(x) = f(-x)$ then the function is called an *odd* function. An odd function is one where the function is unchanged if you rotate it 180° around the point $(0, 0)$. Examples are

$$\begin{aligned} f(x) &= \sin x && \text{since} && f(-x) = \sin(-x) = -\sin x = -f(x), \\ g(x) &= x^3 && \text{since} && g(-x) = (-x)^3 = -x^3 = -g(x). \end{aligned}$$

- You must be able to construct compositions of functions. Note that $f(g(x))$ is not usually the same as $g(f(x))$. For example if $f(x) = x^2$ and $g(x) = x + 1$ then $f(g(x)) = f(x + 1) = (x + 1)^2 = x^2 + 2x + 1$. Contrast this with $g(f(x)) = g(x^2) = x^2 + 1$.

- Sometimes you will be asked to describe a quadratic of the form $ax^2 + bx + c$ in terms of $f(x) = x^2$. It is often useful to *complete the square*. Very quickly I will go through a couple of examples of how to do this:

$x^2 + 10$	$\Rightarrow$	Clearly just $f(x) + 10$.
$x^2 + 6x + 10$	$\Rightarrow$	Complete square to get $(x + 3)^2 - 9 + 10 = (x + 3)^2 + 1$ so it is $f(x + 3) + 1$, which is the translation $\binom{-3}{1}$ of x^2.
$2x^2 + 16x + 1$	$\Rightarrow$	Complete square to get $2(x+4)^2 - 31$ so it is $2f(x+4) - 31$, which is a stretch of factor 2 away from the x-axis, followed by a translation $\binom{-4}{-31}$ of x^2.

- The inverse of a function $f(x)$ is denoted $f^{-1}(x)$. To find the inverse of a function you swap round the x and the y and make y the subject again. This will be the inverse of the original function. For example find the inverse of $f(x) = \sqrt{x^3 + 2}$ gives

$$\begin{aligned} f(x) &= \sqrt{x^3 + 2}, \\ \Rightarrow \qquad y &= \sqrt{x^3 + 2}, \\ \Rightarrow \qquad x &= \sqrt{y^3 + 2}, \\ \Rightarrow \qquad y &= \sqrt[3]{x^2 - 2}, \\ \Rightarrow \qquad f^{-1}(x) &= \sqrt[3]{x^2 - 2}. \end{aligned}$$

- A function only has an inverse if it is a one-to-one mapping. If the original function is a many-to-one function (e.g. $y = x^2$ or any of the trig functions) you must restrict its domain to make it a one-to-one mapping (e.g. for $y = x^2$ restrict domain to $x \geqslant 0$). The domain and range of a function are switched in its inverse. For example if $f(x)$ has domain $x > 8$ and range $f(x) \leqslant -10$, then its inverse $f^{-1}(x)$ has domain $x \leqslant -10$ and range $f^{-1}(x) > 8$.

- Geometrically the relationship between a function and its inverse is a reflection in the line $y = x$. A useful spin-off from this result is that if you are asked to find where a function equals its inverse (i.e. $f(x) = f^{-1}(x)$) all you need to do is solve $f(x) = x$ or $f^{-1}(x) = x$; take your pick.

- Given a point on a function ($(3, 4)$, say) then the equivalent point on its inverse is $(4, 3)$ because it has been reflected in $y = x$. If the gradient at $(3, 4)$ was 7, then the gradient on the inverse will be its reciprocal $\frac{1}{7}$.

Modulus

- The modulus function makes everything you put into it positive. For example $|4| = 4$ and $|-6| = 6$. If something negative is 'fed in' to the mod function then it multiplies it by -1 to turn it positive; otherwise it leaves it alone.

- If you have an expression such as $|x-4|$, then the critical value for x is $x = 4$; if $x > 4$ then the expression is just $x - 4$ and if $x < 4$ then the expression becomes $-x + 4$ because the mod function multiplies it by -1 to turn it positive. This idea helps us solve modulus equations; for example to solve $|2x - 1| = 6$ we first look for the critical values of x; here clearly $x = \frac{1}{2}$. We therefore set up two equations depending on whether $x > \frac{1}{2}$ or $x < \frac{1}{2}$:

$$\begin{aligned} \text{If } x &< \tfrac{1}{2} \\ \text{then } -2x + 1 &= 6 \\ x &= -\tfrac{5}{2}, \end{aligned} \qquad \begin{aligned} \text{If } x &> \tfrac{1}{2} \\ \text{then } 2x - 1 &= 6 \\ x &= \tfrac{7}{2}. \end{aligned}$$

  We perform a little check at the end to check that the solutions found actually satisfy the conditions on x are met; the left hand equation is valid if $x < \frac{1}{2}$ and the solution we have found *is* less that $\frac{1}{2}$; the right hand equation is valid if $x > \frac{1}{2}$ and the solution *is* greater than $\frac{1}{2}$. Therefore both solutions found are valid.

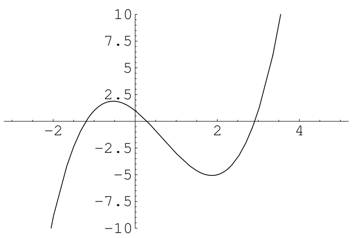
- If you merely have an equation such as |something| = |something else| then just get rid of the mods and square both sides to get (something)^2 = (something else)^2. *Check* your answers back in the original mod equation to check they work!

- Consider the intimidating looking $|2x-1|-1 < |x+2|$. As with most inequalities a good first step is to solve the *equality*; i.e. solve $|2x - 1| - 1 = |x + 2|$. The critical x values are $x = -2$ and $x = \frac{1}{2}$ so we need to set up three different equations depending whether x is $x < -2$, $-2 < x < \frac{1}{2}$ or $x > \frac{1}{2}$ and solve:

$$\begin{aligned} \text{If } x &< -2 \\ \text{then } -2x + 1 - 1 &= -x - 2 \\ x &= 2, \end{aligned} \qquad \begin{aligned} \text{If } -2 < x &< \tfrac{1}{2} \\ \text{then } -2x + 1 - 1 &= x + 2 \\ x &= -\tfrac{2}{3}, \end{aligned} \qquad \begin{aligned} \text{If } x &> \tfrac{1}{2} \\ \text{then } 2x - 1 - 1 &= x + 2 \\ x &= 4. \end{aligned}$$

  Performing our check again we see that two solutions are fine, but $x = 2$ is *not* a solution because the equation was only valid if $x < -2$. Therefore the solution of the equation is $x = -\frac{2}{3}$ or $x = 4$. To solve the inequality we need to see if a number less than $-\frac{2}{3}$ works in the inequality (it doesn't), to see if a number between $-\frac{2}{3}$ and 4 works (it does) and to see if a number greater than 4 works (it doesn't). Therefore $-\frac{2}{3} < x < 4$ is the solution to the question.

- Given a graph of $y = f(x)$ you must be able to draw the graph of $y = |f(x)|$; this is done by leaving any parts of the curve above the x-axis where they are and reflecting parts of the curve under the x-axis so that the are above the x-axis. In the reflected parts, the equation of the curve would be $y = -f(x)$. For example:



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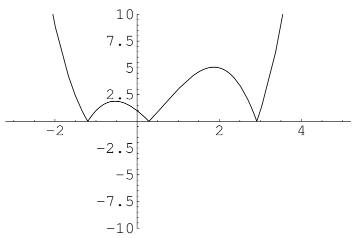

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Trigonometry

- By definition $\sec\theta \equiv \dfrac{1}{\cos\theta}$, $\quad \operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$, $\quad \cot\theta \equiv \dfrac{1}{\tan\theta}$.
- If you get an equation where one of the new trig functions equals a constant, then just take the reciprocal of each side and solve *à la* C2. For example

$$\sec\theta = 5 \qquad \Rightarrow \qquad \frac{1}{\cos\theta} = 5 \qquad \Rightarrow \qquad \cos\theta = \frac{1}{5}.$$

- Know the graphs of $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$. Page 91/2 of your textbook.
- By dividing $\sin^2 x + \cos^2 x \equiv 1$ by $\sin^2 x$ and $\cos^2 x$ we can derive

$$1 + \cot^2 x \equiv \operatorname{cosec}^2 x \quad \text{and} \quad \tan^2 x + 1 \equiv \sec^2 x \quad \text{respectively.}$$

  These create a whole new family of equations that reduce to a quadratic in disguise. For example solve $3\cot^2\theta + 5\operatorname{cosec}\theta + 1 = 0$ in the range $0 \leqslant \theta \leqslant 2\pi$. Firstly note we will need to replace the $\cot^2\theta$ by $\operatorname{cosec}^2\theta - 1$ to reduce the equation to one trig function only.

$$\begin{aligned} 3\cot^2\theta + 5\operatorname{cosec}\theta + 1 &= 0 \\ 3(\operatorname{cosec}^2\theta - 1) + 5\operatorname{cosec}\theta + 1 &= 0 \\ 3\operatorname{cosec}^2\theta + 5\operatorname{cosec}\theta - 2 &= 0 \\ (3\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 2) &= 0 \\ \operatorname{cosec}\theta = \tfrac{1}{3} \quad \text{or} \quad \operatorname{cosec}\theta &= -2. \end{aligned}$$

  Therefore $\sin\theta = 3$ which has no solutions, or $\sin\theta = -\frac{1}{2}$ which gives the solutions $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.
- You must know, and be able to apply, the compound angle formulae:

$$\begin{aligned} \sin(A \pm B) &= \sin A\cos B \pm \cos A\sin B, \\ \cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B, \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}. \end{aligned}$$

- You must know, and be able to apply, the double angle formulae (derived by setting $A = B$ in the compound angle formulae above):

$$\begin{aligned} \sin 2A &= 2\sin A\cos A, \\ \cos 2A &= \cos^2 A - \sin^2 A, \\ &= 2\cos^2 A - 1, \\ &= 1 - 2\sin^2 A, \\ \tan 2A &= \frac{2\tan A}{1 - \tan^2 A}. \end{aligned}$$

  Notice there are three versions of the double angle formula for $\cos 2A$; you need to *think hard* about which form you will need for the question you are solving. You will hardly ever need the fist of the three ($\cos^2\theta - \sin^2\theta$) because it involves two different trig functions; the aim is, usually, to get only one.

- You must be able to convert form the form $a\cos\theta \pm b\sin\theta$ into either $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$; the question will specify which. This then enables us to solve equations of the form

$$a\cos\theta \pm b\sin\theta = \text{constant}.$$

For example express $3\cos\theta - 5\sin\theta$ in the form $R\cos(\theta+\alpha)$. Always start by looking at the coefficients of $\cos\theta$ and $\sin\theta$ in the original expression; here they are 3 and 5 (ignore the sign). Sum their squares and square root (like Pythagoras) and factorise out:

$$3\cos\theta - 5\sin\theta \equiv \sqrt{34}\left[\frac{3}{\sqrt{34}}\cos\theta - \frac{5}{\sqrt{34}}\sin\theta\right].$$

Next consider the form of the answer we are aiming for; here "$R\cos(\theta+\alpha)$". The expansion of "$R\cos(\theta+\alpha)$" is "$R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$". Comparing

$$\sqrt{34}\left[\frac{3}{\sqrt{34}}\cos\theta - \frac{5}{\sqrt{34}}\sin\theta\right] \quad \text{with} \quad R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

we see instantly $R = \sqrt{34}$. We also require $\frac{3}{\sqrt{34}} = \cos\alpha$ and $\frac{5}{\sqrt{34}} = \sin\alpha$; solving either of those two we find $\alpha = 59.0°$ (to 1 d.p). Therefore

$$3\cos\theta - 5\sin\theta \equiv \sqrt{34}\cos(\theta + 59.0°).$$

- The trig functions all have inverses if we restrict the domain. The conventional restrictions to allow inversion are

Function	Domain	Domain
$y=\sin x$	$-90° \leqslant x \leqslant 90°$	$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$
$y=\cos x$	$0° \leqslant x \leqslant 180°$	$0 \leqslant x \leqslant \pi$
$y=\tan x$	$-90° < x < 90°$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

Know what the graphs of $y=\sin^{-1}x$, $y=\cos^{-1}x$ and $y=\tan^{-1}x$ look like.

Exponentials & Logarithms

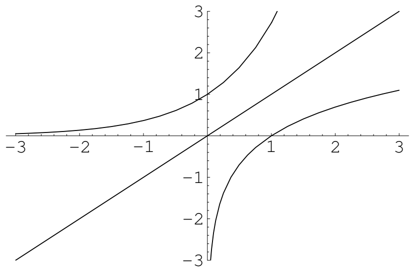
- Know that e is a special number in mathematics. It is approximately 2.7182818284... and it is irrational (i.e. it can't be expressed as a fraction; similar to π).
- If the base of a logarithm is e then we call it a 'natural logarithm'. Written $\log_e x \equiv \ln x$.
- We already know that logarithms and exponentials are inverses of each other with the relationships

$$\log_{10}(10^x) \equiv x \qquad \text{and} \qquad 10^{\log_{10} x} \equiv x.$$

The same is true for natural logarithms and exponents of e;

$$\ln(e^x) \equiv x \qquad \text{and} \qquad e^{\ln x} \equiv x.$$

- Below is a graph of $y = e^x$ and $y = \ln x$ showing the inverse relationship between the two (reflecting in $y = x$):



This also shows that you can't 'ln' a negative number and that $\ln 1 = 0$.

- All the laws of logarithms from C2 are true for natural logarithms (e.g. $\ln ab = \ln a + \ln b$). For example make a the subject of the following equation (a few steps missed out):

$$\begin{aligned}\ln(a-1) - \ln(a+1) &= b\\ \ln\left(\frac{a-1}{a+1}\right) &= b\\ \frac{a-1}{a+1} &= e^b\\ a(1-e^b) &= 1+e^b\\ a &= \frac{1+e^b}{1-e^b}.\end{aligned}$$

- You must understand that many physical systems can be modelled by either exponential growth or exponential decay. The most general form is $y = a \times b^x$. If $b > 1$ then the curve represents *exponential growth*. If $b < 1$ then the curve represents *exponential decay*. For example if the number of swine flu sufferers is modelled by $N = 5 \times 7^t$, where t is time measured in days, then find the amount of time for 2 billion people to have caught the disease. We need to solve $2 \times 10^9 = 5 \times 7^t$. So

$$\frac{2\times 10^9}{5} = 7^t \quad \Rightarrow \quad \log\left(\frac{2\times 10^9}{5}\right) = t\log 7 \quad \Rightarrow \quad t = 10.2 \text{ days! (to 3 s.f.)}.$$

(Cue dramatic music…)

- Any exponential relationship $y = a \times b^x$ can be converted to an exponential form using e. This is useful because to differentiate exponential relationships they have to be of the form $y = a \times e^{kt}$. This is done using the powerful statement that (something $\equiv e^{\ln \text{something}}$), so

$$\begin{aligned}y &= a \times b^x\\ y &= a \times e^{\ln(b^x)}\\ y &= a \times e^{x\ln b} && \text{(by 'log law' } \log(a^n) = n\log a)\\ y &= a \times e^{kx}, && \text{(where } k = \ln b).\end{aligned}$$

- An exponential can never equal zero (see graph above). Therefore if you have an equation with lots of exponential 'bits' that you can factorise out, then you are allowed to divide through (in a way that is forbidden with trig functions). For example if $2x^2e^{2x} + 3xe^{2x} - 2e^{2x} = 0$, factorise out the e^{2x} to get $e^{2x}(2x^2 + 3x - 2) = 0$. Divide by e^{2x} to get $2x^2 + 3x - 2 = 0$ which solves to $x = \frac{1}{2}$ or $x = -2$.

- To differentiate an exponential the basic building block is

$$y = e^x \qquad \Rightarrow \qquad \frac{dy}{dx} = e^x.$$

That is *why* 'e' is so important; it gives us the exponential that differentiates to itself. Combined with the chain/product/quotient rule (below) we can build on this starting point. (Some students think that if $y = e^x$, then $\frac{dy}{dx} = xe^{x-1}$. Do not be one of them! Exponentials are fundamentally different to polynomials.)

- To differentiate a natural logarithm the basic building block is

$$y = \ln x \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{x}.$$

  Again combined with the chain/product/quotient rule (below) we can build on this starting point.

Differentiation

- The basic building blocks for differentiation (that we know at present) are:

$$\begin{array}{ccc} y = ax^n & y = e^x & y = \ln x \\ \dfrac{dy}{dx} = anx^{n-1}, & \dfrac{dy}{dx} = e^x, & \dfrac{dy}{dx} = \dfrac{1}{x}. \end{array}$$

  Also we know the idea that (for $f(x) \equiv f$, $g(x) \equiv g$ and k = constant)

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g) \quad \text{and} \quad \frac{d}{dx}(kf) = k\frac{d}{dx}(f).$$

  (In big-boy speak we say that $\frac{d}{dx}$ is a linear operator.)

- The *chain rule* is incredibly important! It states that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

  This seems obvious from the way that differentials are written, but remember that they should not be thought of as fractions.

- If a bit of a $y = \dots$ is making the differentiation difficult, then ask yourself the question "would making the complicated bit u make it easier for me to deal with?" For example with $y = (2x-5)^{20}$ the function would be considerably easier if $u = 2x - 5$ because y becomes $y = u^{20}$. Similarly with $y = e^{x^2+1}$ my life would be easier if $u = x^2 + 1$ because y would become $y = e^u$.

- It can be applied as follows to the example $y = (x^4 + x)^{10}$. Let $u = x^4 + x$, so

$$\begin{array}{cc} y = u^{10} & u = x^4 + x \\ \dfrac{dy}{du} = 10u^9 & \dfrac{du}{dx} = 4x^3 + 1. \end{array}$$

  Therefore $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 10u^9 \times (4x^3 + 1) = 10(4x^3 + 1)(x^4 + x)^9$.

- The above method works all the time but it is a little slow. You will notice the general result that if $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$. So we can just write down the answer to similar problems. For example if $y = (3x^2 + 1)^5$ then $\frac{dy}{dx} = 30x(3x^2 + 1)^4$.

- We can also combine the chain rule with exponentials and logarithms to gain the following important results:

$$\begin{aligned} \frac{d}{dx}(e^{ax}) &= ae^{ax} && \text{using } u = ax \\ \frac{d}{dx}(e^{f(x)}) &= f'(x)e^{f(x)} && \text{using } u = f(x) \\ \frac{d}{dx}(\ln ax) &= \frac{a}{ax} = \frac{1}{x} && \text{using } u = ax \\ \frac{d}{dx}(\ln f(x)) &= \frac{f'(x)}{f(x)} && \text{using } u = f(x). \end{aligned}$$

- The *product rule* states that when $y = u \times v$ (where u and v are functions of x) we can differentiate it using the product rule. It states that

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

For example if $y = x^2(x^3 - 1)^3$ then

$$\begin{aligned} \frac{dy}{dx} &= [2x \times (x^3 - 1)^3] + [x^2 \times 3(x^3 - 1)^2 \times 3x^2] \\ &= 2x(x^3 - 1)^3 + 9x^4(x^3 - 1)^2 \\ &= x(x^3 - 1)^2[2(x^3 - 1) + 9x^3] \\ &= x(x^3 - 1)^2(11x^3 - 2). \end{aligned}$$

- With the product rule you often end up with expressions such as

$$\frac{dy}{dx} = 2x^3(2x + 1)^{-4} - x^4(2x + 1)^{-5}.$$

When tidying these things up you must pull out (as always) *the lowest power* of any common elements even if they are negative or fractional; here we have x^3 and $(2x + 1)^{-5}$:

$$\begin{aligned} \frac{dy}{dx} &= 2x^3(2x + 1)^{-4} - x^4(2x + 1)^{-5} \\ &= x^3(2x + 1)^{-5}[2(2x + 1) - x] \\ &= \frac{x^3(3x + 2)}{(2x + 1)^5}. \end{aligned}$$

- Very similar to the product rule is the *quotient rule*. It is used for functions of the form $y = \frac{u}{v}$. It states

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

For example differentiating $y = \frac{x^3}{x^2+1}$ gives

$$\frac{dy}{dx} = \frac{(x^2 + 1) \times 3x^2 - x^3 \times 2x}{(x^2 + 1)^2} = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2}.$$

- Once again, although $\frac{dy}{dx}$ is not a fraction, it can be treated as such when taking its reciprocal, so

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}.$$

For example if you have $V = \frac{4}{3}\pi r^3$ then $\frac{dV}{dr} = 4\pi r^2$ and also $\frac{dr}{dV} = \frac{1}{4\pi r^2}$. This idea most useful in the topic of…

- …*connected rates of change*. Here you need to use the chain rule to 'connect' differentials you know to get one you need. Questions mostly ask you for $\frac{dy}{dx}$ (say) and you need to find a third variable to construct $\frac{dy}{dx} = \frac{dy}{d\ldots} \times \frac{d\ldots}{dx}$ by the chain rule. For example: The area A of a circle is increasing a rate of 3cm^2/s, find the rate at which the radius r is increasing when $r = 20$cm. We want to find $\frac{dr}{dt}$ so

$$\begin{aligned}\frac{dr}{dt} &= \frac{dr}{dA} \times \frac{dA}{dt} \qquad \text{but} \qquad A = \pi r^2, \text{ so } \frac{dA}{dr} = 2\pi r.\\ &= \frac{1}{2\pi r} \times 3\\ &= \frac{3}{40\pi}.\end{aligned}$$

Integration

- The central idea of calculus is that integration and differentiation are the inverse operations of each other in the same way that plus is the inverse operation of subtraction. In C3 a favourite type of question is to differentiate something using the above rules and then integrate something similar later in the question. View the question as a whole!
- Our basic building blocks for integration are therefore

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c, \qquad \int e^x\,dx = e^x + c, \qquad \int \frac{1}{x}\,dx = \ln x + c.$$

- A big result is gained by inspection below, but worth stating alone:

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c.$$

- Integration by *inspection* is effectively "spotting the answer" by an intermediate guess. The intermediate guess is then differentiated mentally and the final answer should then only be a constant factor out. Things to note are that the power on $e^{\text{something}}$ never changes and be on the lookout for integrals where the top line is *almost* the derivative of the bottom line. Here are a few examples:

Integral	Guess	Answer
$\int (2x+3)^{15}\,dx$	$(2x+3)^{16} + c$	$\frac{1}{32}(2x+3)^{16} + c$,
$\int (1-3x)^{-4}\,dx$	$(1-3x)^{-3} + c$	$\frac{1}{9}(1-3x)^{-3} + c$,
$\int 2\sqrt{4x+1}\,dx$	$(4x+1)^{\frac{3}{2}} + c$	$\frac{1}{3}(4x+1)^{\frac{3}{2}} + c$,
$\int 2e^{3x-5}\,dx$	$e^{3x-5} + c$	$\frac{2}{3}e^{3x-5} + c$,
$\int 7xe^{x^2+1}\,dx$	$e^{x^2+1} + c$	$\frac{7}{2}e^{x^2+1} + c$,
$\int \frac{7}{1-4x}\,dx$	$\ln(1-4x) + c$	$-\frac{7}{4}\ln(1-4x) + c$,
$\int \frac{e^{2x}}{1-e^{2x}}\,dx$	$\ln(1-e^{2x}) + c$	$-\frac{1}{2}\ln(1-e^{2x}) + c$.

You must practice this a lot…it only comes easily after a while. [Since most students also take C3 and C4 at the same time it is worth noting that all the above can be done by the C4 technique of *integration by substitution*.]

- $\int_a^b \pi y^2\,dx$ is the volume of revolution of the curve y rotated about the x-axis between $x = a$ and $x = b$. All that is needed for you to do is calculate y^2 in terms of x from y. For example find the volume of revolution of the solid formed by rotating the curve $y = \sqrt{2x+3}$ about the x-axis between $x = 10$ and $x = 14$. We need to evaluate $\int_a^b \pi y^2\,dx = \int_{10}^{14} \pi y^2\,dx$. Now the curve is $y = \sqrt{2x+3}$ so to find y^2 in terms of x we need only square the equation $\Rightarrow y^2 = 2x + 3$. We therefore evaluate

$$\int_{10}^{14} \pi y^2\,dx = \pi \int_{10}^{14} (2x+3)\,dx = \pi \left[x^2 + 3x\right]_{10}^{14} = 108\pi.$$

- For volumes of revolution around the y-axis switch the x and the y and use $\int_p^q \pi x^2\,dy$ between $y = p$ and $y = q$. For example find the volume of revolution of the solid formed by rotating the line $y = 3x-2$ about the y-axis between $y = 0$ and $y = 5$. We need to evaluate $\int_p^q \pi x^2\,dy = \int_0^5 \pi x^2\,dy$. Now the line is $y = 3x - 2$ so to find x^2 in terms of y, we make x the subject and square;

$$y = 3x - 2 \quad \Rightarrow \quad x = \frac{y+2}{3} \quad \Rightarrow \quad x^2 = \frac{y^2+4y+4}{9}.$$

We therefore evaluate

$$\int_0^5 \pi x^2\,dy = \pi \int_0^5 \left(\frac{y^2+4y+4}{9}\right) dy = \frac{\pi}{9}\left[\frac{y^3}{3} + 2y^2 + 4y\right]_0^5 = \frac{335\pi}{27}.$$

Numerical Methods

- Given an equation $f(x) = g(x)$ it is often not possible to solve them *analytically* (by algebraic manipulation) and we are forced to use numerical methods that 'home in' on the solution. You need to know two for C3: "search for a change of sign" and "fixed point iteration".

- *Search for a change of sign* 'homes in' on a solution to an equation by sandwiching the solution between two numbers. Those two numbers can gradually be brought together to improve knowledge of where the solution is. Given an equation ($e^x = 15x + 3$, say) it is best to get one side equal to zero ($0 = e^x - 15x - 3$). Then *define* $f(x) = e^x - 15x - 3$. Then put values into $f(x)$ and look for a change of sign.

$$
\begin{array}{ll}
f(-1) = 12.36787944\ldots & +\text{ve} \\
f(0) = -2 & -\text{ve} \\
f(1) = -15.28171817\ldots & -\text{ve} \\
f(2) = -25.6109439\ldots & -\text{ve} \\
f(3) = -27.91446308\ldots & -\text{ve} \\
f(4) = -8.401849967\ldots & -\text{ve} \\
f(5) = 70.4131591\ldots & +\text{ve}
\end{array}
$$

From this we can see that there are two solutions (α and β) such that

$$-1 < \alpha < 0 \qquad \text{and} \qquad 4 < \beta < 5.$$

If you were interested in finding β to 2 decimal places (say) then you would next evaluate

$$f(4.1), f(4.2), \ldots, f(4.9)$$

and you should discover $4.1 < \beta < 4.2$. Next

$$f(4.11), f(4.12), \ldots, f(4.19)$$

and you should discover $4.18 < \beta < 4.19$. You should resist the temptation (however strong) to state $\beta = 4.18$ (to 2 d.p.) as your final answer. It is still possible that the answer could still be $\beta = 4.19$ (to 2 d.p.). You must check 4.185 and then think! *Hard*!

We find $f(4.185) < 0$, so the change of sign exists between 4.185 and 4.19 so final stated answer should be $\beta = 4.19$ (to 2 d.p.)

- *Fixed point iteration* works by taking an equation and rearranging to isolate an x in the form $x = g(x)$. From this rearrangement we form an iterative formula

$$x_{n+1} = g(x_n).$$

It is important to note that there exist many possible rearrangements of an equation; for the equation $x^3 - 3x + 4 = 0$ here are a few:

$$x = \sqrt[3]{3x-4} \qquad x = \frac{x^3+4}{3} \qquad x = \frac{3x-4}{x^2}.$$

However, the exam will usually specify which one they want[7]. In the above example let's use the first one and create $x_{n+1} = \sqrt[3]{3x_n - 4}$. The starting value for the iteration is denoted x_0 (or x_1) and you should either use the value specified in the question or choose a value close to where you know the solution exists[8]. Here let's use $x_0 = -1$.

To save time, you can use your calculator to speed up the process a lot. First type "-1 =" to enter -1 as the "Ans" on your calculator. Then type "$\sqrt[3]{(3 \times \text{Ans} - 4)}$". Press "=" repeatedly to see the results of the iteration. You should find:

$$\begin{aligned}
x_0 &= -1\\
x_1 &= -1.912931183\\
x_2 &= -2.134410543\\
x_3 &= -2.18324263\\
x_4 &= -2.19321102\\
x_5 &= -2.19528142\\
&\vdots \qquad \cdots\text{keep pressing "=" lots and eventually}\cdots\\
x &= -2.19582 \text{ to (5 d.p.)}
\end{aligned}$$

Always state the accuracy to which you give your answer (sig figs or d.p.s). If when you keep pressing "=" it settles to one number we say the iteration *converges*; otherwise it *diverges*.

- Sometimes a question gives you an iteration and asks for the equation which has been solved (or to show that the number the iteration converges to represents a solution of another given equation). All you do is remove the $n+1$ and n subscripts and rearrange: For example

$$\begin{aligned}
x_{n+1} &= \ln(\sqrt[3]{1-2x_n})\\
x &= \ln(\sqrt[3]{1-2x})\\
e^x &= \sqrt[3]{1-2x}\\
e^{3x} + 2x - 1 &= 0.
\end{aligned}$$

[7]*rant* It is worth noting just how unrealistic this situation is; in practice you will discover that some of these rearrangements work a treat and some of them fail miserably. This should be part of a coursework (*à la* MEI) and not part of an exam! *rant*

[8]If the first part of a question gets you to show the solution exists between 2.1 and 2.2 then start with $x_0 = 2.1$

Simpson's Rule

- Similar to the trapezium rule is *Simpson's Rule*. It can be used to approximate integrals. It uses a quadratic curve to approximate the curve rather than a straight line, and is therefore rather more accurate. Unlike the trapezium rule it is hard to say whether the approximation will be an over or under-estimate. Therefore you don't get questions on it.
- In class I refer to "Simpson Chunks"; this is a Stone-ism you will not hear elsewhere. One "Simpson Chunk" contains two intervals/strips and three ordinates.

  In general

  $$n \text{ "Simpson Chunks"} \quad \Leftrightarrow \quad 2n \text{ Intervals/Strips} \quad \Leftrightarrow \quad 2n+1 \text{ Ordinates.}$$

  The heights on the ordinates are the y-values of the curve. They are labelled $y_0, y_1, \ldots, y_{2n}$. **Never** forget that the first height on the left is denoted y_0 and **not** y_1; if you do the whole question will go wrong because your 'odds' and 'evens' will be wrong!
- Simpson's Rule states (where h is the distance between each ordinate/height):

$$\int_a^b y\,dx \approx \frac{h}{3}\left[y_0 + y_{2n} + 4(y_1 + y_3 + \cdots + y_{2n-1}) + 2(y_2 + y_4 + \cdots + y_{2n-2})\right].$$

$$\int_a^b y\,dx \approx \frac{h}{3}\left[\text{'sum ends'} + 4(\text{'sum internal odds'}) + 2(\text{'sum internal evens'})\right].$$

- For example use 8 intervals to approximate $\int_{-4}^{4} \frac{1}{1+x^2}\,dx$. Each interval must have width 1 since the total width is 8. There must be nine ordinates. A table for the ordinates:

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
$\frac{1}{17}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$

  Therefore

$$\int_{-4}^{4} \frac{1}{1+x^2}\,dx \approx \frac{1}{3}\left[\frac{1}{17} + \frac{1}{17} + 4\left(\frac{1}{10} + \frac{1}{2} + \frac{1}{2} + \frac{1}{10}\right) + 2\left(\frac{1}{5} + 1 + \frac{1}{5}\right)\right]$$
$$\approx 2.573 \text{ (to 3 d.p.).}$$

Algebra

- Review binomial expansion from C2 for $(x+y)^n$ for positive integer n. Notice that it is valid for *any* x and y and that the expansion has $n+1$ terms.
- The general binomial expansion is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

  and is valid for any n (fractional or negative) but $-1 < x < 1$ (i.e. $|x| < 1$). Notice also it must start with a 1 in the brackets. For example expand $(4-x)^{-1/2}$.

$$\begin{aligned}
(4-x)^{-1/2} &= \left(4\left(1-\frac{x}{4}\right)\right)^{-1/2}\\
&= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}\\
&= \frac{1}{2}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{x}{4}\right)^2+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{x}{4}\right)^3+\cdots\right]\\
&= \frac{1}{2}\left[1+\frac{x}{8}+\frac{3x^2}{128}+\frac{15x^3}{3072}+\cdots\right]=\frac{1}{2}+\frac{x}{16}+\frac{3x^2}{256}+\frac{15x^3}{6144}+\cdots.
\end{aligned}$$

It is only valid for $|x/4|<1 \quad \Rightarrow \quad |x|<4$.

- Another example: Find first 3 terms in the expansion for $\dfrac{(3+x)^2}{1+\frac{x}{2}}$.

$$(3+x)^2\left(1+\frac{x}{2}\right)^{-1}=(9+6x+x^2)\left(1-\frac{x}{2}+\frac{x^2}{4}+\ldots\right)=9+\frac{3}{2}x+\frac{1}{4}x^2+\ldots.$$

- You must be able to simplify algebraic fractions; best tactic is always to factorise and cancel:

$$\frac{4x^5-10x^4-6x^3}{12x^6-18x^5-12x^4}=\frac{2x^3(2x^2-5x-3)}{6x^4(2x^2-3x-2)}=\frac{2x^3(2x+1)(x-3)}{6x^4(x-2)(2x+1)}=\frac{x-3}{3x(x-2)}.$$

- You must be able to divide a polynomial ($p(x)$) by a divisor ($a(x)$), finding the quotient ($q(x)$) and remainder ($r(x)$). If there is no remainder then $a(x)$ is a factor of $p(x)$. It is always such that

$$\frac{p(x)}{a(x)}=q(x)+\frac{r(x)}{a(x)} \qquad \Rightarrow \qquad p(x)=a(x)q(x)+r(x).$$

The order of $q(x)$ is the order of $p(x)$ subtract the order of $a(x)$. The order of $r(x)$ is *at most* one less than $a(x)$. For example if you have a quintic (power 5 polynomial) divided by a quadratic you would expect

$$\begin{aligned}
\frac{\text{quintic}}{\text{quadratic}} &= \text{cubic}+\frac{\text{linear}}{\text{quadratic}},\\
\frac{\text{quintic}}{\text{quadratic}} &= Ax^3+Bx^2+Cx+D+\frac{Ex+F}{\text{quadratic}}.
\end{aligned}$$

Of course it *may* turn out that $Ex+F$ is just a constant or zero (if the cubic divides the quintic).

- Division is most easily done step-by-step working *down* the powers of $p(x)$. For example divide $2x^4-x^3+3x^2-7x+1$ by x^2+2x+3:

$$\begin{aligned}
2x^4-x^3+3x^2-7x+1 &= (x^2+2x+3)(\text{quadratic})+(\text{remainder})\\
&= (x^2+2x+3)(2x^2+\ldots)+(\text{remainder}) && x^4\checkmark\\
&= (x^2+2x+3)(2x^2-5x+\ldots)+(\text{remainder}) && x^3\checkmark\\
&= (x^2+2x+3)(2x^2-5x+7)+(\text{remainder}) && x^2\checkmark\\
&= (x^2+2x+3)(2x^2-5x+7)-6x+\text{const.} && x\checkmark\\
&= (x^2+2x+3)(2x^2-5x+7)-6x-20 && \text{const.}\checkmark
\end{aligned}$$

Therefore $q(x)=2x^2-5x+7$ and $r(x)=-6x-20$.

- Partial fractions is effectively the reverse of combining together two algebraic fractions. For example

$$\frac{1}{x+1}+\frac{1}{x+2} \quad \begin{matrix}\longrightarrow \\ \longleftarrow\end{matrix} \quad \begin{matrix}\text{Algebraic Fractions} \\ \text{Partial Fractions}\end{matrix} \quad \begin{matrix}\longrightarrow \\ \longleftarrow\end{matrix} \quad \frac{2x+3}{(x+1)(x+2)}.$$

  You can use partial fractions provided the order of the top line is less than the order of the bottom line.

- If you simply have a product of linear factors on the bottom line then you split out into that many terms with constants placed on top (usually denoted by A, B, C, etc.). Place this equal (in an identity "≡") to the original expression and then multiply through to get rid of the denominators. For example:

$$\begin{aligned} \frac{7x-1}{(2x+1)(x-1)} &\equiv \frac{A}{2x+1}+\frac{B}{x-1} \\ \Rightarrow \quad 7x-1 &\equiv (x-1)A+(2x+1)B. \end{aligned}$$

  Because this is an identity we can choose any value of x we fancy to help us discover A and B. In this case letting $x = 1$ is a good choice because one of the brackets become zero. Similarly $x = -\frac{1}{2}$ is another good choice[9]; we put these into the identity.

$$\begin{aligned} x = 1 \quad &\Rightarrow \quad 7-1 \equiv 3B \quad \Rightarrow \quad \underline{B=2}, \\ x = -\tfrac{1}{2} \quad &\Rightarrow \quad -\tfrac{7}{2}-1 \equiv -\tfrac{3}{2}A \quad \Rightarrow \quad \underline{A=3}. \end{aligned}$$

  Therefore $\frac{7x-1}{(2x+1)(x-1)} \equiv \frac{3}{2x+1}+\frac{2}{x-1}$.

- If you have a repeated factor in the denominator then you deal with it as follows (notice the top line is a quadratic and the bottom a cubic, so partial fractions are fine):

$$\begin{aligned} \frac{5x^2-10x+1}{(x-3)(x-1)^2} &\equiv \frac{A}{x-3}+\frac{B}{x-1}+\frac{C}{(x-1)^2} \\ \Rightarrow \quad 5x^2-10x+1 &\equiv (x-1)^2A+(x-3)(x-1)B+(x-3)C. \end{aligned}$$

  Similarly, good values of x to choose are $x = 1$ and $x = 3$:

$$\begin{aligned} x = 1 \quad &\Rightarrow \quad 5-10+1 \equiv -2C \quad \Rightarrow \quad \underline{C=2}, \\ x = 3 \quad &\Rightarrow \quad 45-30+1 \equiv 4A \quad \Rightarrow \quad \underline{A=4}. \end{aligned}$$

  This tells us that

$$5x^2-10x+1 \equiv 4(x-1)^2+(x-3)(x-1)B+2(x-3),$$

  but it hasn't told us B. I would consider $x = 0$, here, and sub in to discover

$$0-0+1 \equiv 4+3B-6 \Rightarrow \underline{B=1}.$$

  Therefore $\frac{5x^2-10x+1}{(x-3)(x-1)^2} \equiv \frac{4}{x-3}+\frac{1}{x-1}+\frac{2}{(x-1)^2}$.

[9] If you don't spot (or run out of) clever values to use, fret not! Just put in some other numbers (usually something 'nice' like $x = 0$, $x = 1$ or $x = -1$) and solve the resulting equations. If your girlfriend's lucky number is 53 and your mistress's lucky number is 178 then feel free to use those if you wish, although I wouldn't advise it; choose simple numbers!

- Partial fractions are often very useful in evaluating integrals. If you see a quadratic in the bottom line of an integral, then one option your brain should turn to is "can this be split into partial fractions?". For example in this integral, it can be split and then integrated:

$$\begin{aligned}\int \frac{x+10}{x^2+5x+4}\,dx &= \int \frac{x+10}{(x+1)(x+4)}\,dx\\ &= \int \frac{3}{x+1} - \frac{2}{x+4}\,dx\\ &= 3\ln(x+1) - 2\ln(x+4) + c.\end{aligned}$$

Differentiation & Integration

- Know the contents of the formula booklet well. Very well! Lots of problems can be solved simply by looking at the table of differentials and integrals and knowing that integration 'undoes' a differentiation. Some questions get you to differentiate something and *then* get you to integrate something similar. *Always view the question as a whole!*

- When using radians we can differentiate the trigonometric functions. The results are as follows:

$$\begin{aligned} y &= \sin x & y &= \cos x & y &= \tan x\\ \frac{dy}{dx} &= \cos x, & \frac{dy}{dx} &= -\sin x, & \frac{dy}{dx} &= \sec^2 x.\end{aligned}$$

  One can derive the third result from the other two using the quotient rule and that $\tan x \equiv \frac{\sin x}{\cos x}$.

- You can also use these results along with the chain rule to differentiate functions like the following; $y = \sin(x^2+1)$ by letting $u = x^2+1$ and $y = (\tan x)^{10}$ by letting $u = \tan x$.

$$\begin{aligned} y &= \sin(x^2+1) & y &= (\tan x)^{10}\\ \frac{dy}{dx} &= 2x\cos(x^2+1), & \frac{dy}{dx} &= 10\sec^2 x(\tan x)^9.\end{aligned}$$

- Integration by substitution is a way of integrating by replacing the variable given to you (usually x) and replacing it by another (usually u). These days the substitution you are to use is given to you in the exam, but practice will get you better at spotting what to substitute (usually the most complicated term in the integration or the denominator of a fraction). For example $\int x^3(x^4+1)^7\,dx$ we should use $u = x^4+1$.

$$\begin{aligned} &\int x^3(x^4+1)^7\,dx & u &= x^4+1\\ =&\int x^3u^7\,dx & \frac{du}{dx} &= 4x^3\\ =&\int x^3u^7\,\frac{du}{4x^3} & \frac{du}{4x^3} &= dx\\ =&\frac{1}{4}\int u^7\,du & &\\ =&\frac{u^8}{32} + c = \frac{(x^4+1)^8}{32} + c. & &\end{aligned}$$

  We have effectively *"used and abused" u* to help us to get the answer. (Note: I have been *very* sloppy in the above integration because I have mixed my x and u variables; you shouldn't really do this, but it makes the process of conversion clearer.)

- When dealing with definite integrals we need to also convert the limits of the integration and there is no need to convert back to x at the end since all definite integrals are merely numbers. For example

$$\begin{aligned}
&\int_3^4 2x\sqrt{x^2-4}\,dx && u = x^2 - 4 && x = 3 \Rightarrow u = 5\\
&= \int_5^{12} 2xu^{1/2}\,\frac{du}{2x} && \frac{du}{dx} = 2x && x = 4 \Rightarrow u = 12\\
&= \int_5^{12} u^{1/2}\,du && \frac{du}{2x} = dx\\
&= \left[\frac{2}{3}u^{3/2}\right]_5^{12}\\
&= 20.3 \text{ (3sf)}.
\end{aligned}$$

- Know the result $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$.
- We know that if $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$. Therefore by reversal we find

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c.$$

  For example[10]

$$\int x^3e^{x^4}\,dx = \frac{1}{4}\int 4x^3e^{x^4}\,dx = \frac{1}{4}e^{x^4} + c.$$

- Know that $\int \frac{1}{x}\,dx = \ln x + c$.
- We know (by the chain rule) that if $y = \ln(f(x))$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$. Therefore by reversal we find

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$$

  Be on the lookout for expressions where the top line is almost the derivative of the bottom line. For example[11]

$$\int \frac{x^3}{x^4+1}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4+1}\,dx = \frac{1}{4}\ln|x^4+1| + c.$$

- Know the results

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + c \qquad \text{and} \qquad \int \sin ax\,dx = -\frac{1}{a}\cos ax + c.$$

- Always be on the look out for integrals involving a mixture of trigonometric functions. These are usually handled by means of a substitution. For example $\int \cos x(\sin x)^7\,dx$ is best handled by $u = \sin x$ to give $\frac{1}{8}(\sin x)^8 + c$.
- Also know the useful results (all derived from reverse chain rule)

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c \qquad \text{and} \qquad \int f'(x)\sin f(x)\,dx = -\cos f(x) + c.$$

  For example $\int x^3\cos(x^4)\,dx = \frac{1}{4}\sin(x^4) + c$.

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[10] This could also have been evaluated (more slowly) by a substitution of $u = x^4$ which would then have reduced to $\int x^3e^{x^4}\,dx = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^{x^4} + c$.

[11] Again, this could also have been evaluated by the substitution $u = x^4 + 1$.

- When an integral is made up of two 'bits' then we can sometimes use *integration by parts*. It states
$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$
So you will need to decide which 'bit' of the integral you will need to differentiate and which 'part' to integrate. For example in $\int x\sin x\,dx$ it is quite clear that we will need to differentiate the x 'part' and integrate the $\sin x$ 'part'.
$$\begin{aligned}\int x\sin x\,dx &= -x\cos x - \int -\cos x\,dx\\ &= -x\cos x + \sin x + c.\end{aligned}$$

- Another example (this time a definite integral)
$$\begin{aligned}\int_0^2 xe^{2x}\,dx &= \left[\tfrac{1}{2}xe^{2x}\right]_0^2 - \int_0^2 \tfrac{1}{2}e^{2x}\,dx\\ &= \left[\tfrac{1}{2}xe^{2x}\right]_0^2 - \left[\tfrac{1}{4}e^{2x}\right]_0^2\\ &= \left(e^4 - 0\right) - \left(\tfrac{e^4}{4} - \tfrac{1}{4}\right) = \tfrac{3e^4}{4} + \tfrac{1}{4}.\end{aligned}$$

- Initially $\int \ln x\,dx$ looks nothing like it has anything to do with integration by parts because it only has one 'part'. However if we write $\ln x$ as $1\times\ln x$ we can integrate the 1 and differentiate the $\ln x$:
$$\int \ln x\,dx = \int 1\ln x\,dx = x\ln x - \int x.\tfrac{1}{x}\,dx = x\ln x - x + c.$$
This principle can be extended to integrals of the type $\int x^n \ln x\,dx$:
$$\int x^n\ln x\,dx = \tfrac{x^{n+1}}{n+1}\ln x - \int \tfrac{x^{n+1}}{n+1}.\tfrac{1}{x}\,dx = \tfrac{x^{n+1}}{n+1}\ln x - \tfrac{x^{n+1}}{(n+1)^2} + c.$$

- Very occasionally you will need to integrate by parts *twice* to get the final answer. This will almost always be of the form $\int kx^2(\text{something})\,dx$. For example find $\int x^2e^{2x}\,dx$:
$$\begin{aligned}\int x^2e^{2x}\,dx &= \frac{x^2}{2}e^{2x} - \left(\int xe^{2x}\,dx\right)\\ &= \frac{x^2}{2}e^{2x} - \left(\frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx\right)\\ &= \frac{x^2}{2}e^{2x} - \left(\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right) + c\\ &= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + c.\end{aligned}$$

- For the cases of $\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$ we need to recall two forms of the double angle formula for $\cos 2x$: Namely $\cos 2x = 1 - 2\sin^2 x$ (for $\int \sin^2 x\,dx$) and $\cos 2x = 2\cos^2 x - 1$ (for $\int \cos^2 x\,dx$). Re-arranging them both we find:
$$\begin{aligned}\int \sin^2 x\,dx &= \int (\tfrac{1}{2} - \tfrac{1}{2}\cos 2x)\,dx = \tfrac{x}{2} - \tfrac{1}{4}\sin 2x + c,\\ \int \cos^2 x\,dx &= \int (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\,dx = \tfrac{x}{2} + \tfrac{1}{4}\sin 2x + c.\end{aligned}$$
Learn the technique rather than the result!

Implicit Functions

- Given a function in the form $y = f(x)$ we can differentiate it. Implicit differentiation allows us to differentiate a function without making y the subject first. It uses the chain rule that
$$\frac{d\,f(y)}{dx} = \frac{d\,f(y)}{dy} \times \frac{dy}{dx}.$$
So all you do is differentiate the y bits with respect to y and then multiply by $\frac{dy}{dx}$. For example differentiate $y^4 + x^4 = \sin y$ with respect to x. This gives
$$4y^3\frac{dy}{dx} + 4x^3 = \cos y\frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{4x^3}{\cos y - 4y^3}.$$
You must be on the lookout for products in terms of x and y; for example $2xy = e^{2y}$ would differentiate to
$$2x\frac{dy}{dx} + 2y = 2e^{2y}\frac{dy}{dx} \quad \text{so} \quad \frac{dy}{dx} = \frac{2y}{2e^{2y} - 2x} = \frac{y}{e^{2y} - x}.$$
- Another example; find all the stationary points on the curve $x^2 + y^2 + xy = 3$. Differentiating w.r.t. x we find
$$2x + 2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{2x + y}{2y + x}.$$
Stationary points are where $\frac{dy}{dx} = 0$ so solve
$$0 = -\frac{2x + y}{2y + x} \qquad \Rightarrow \qquad y = -2x.$$
Substituting this *back into the original equation* we find
$$x^2 + (-2x)^2 + x(-2x) = 3 \qquad \Rightarrow \qquad x = \pm 1 \qquad \Rightarrow \qquad \text{Points are } (1, -2) \text{ and } (-1, 2).$$
- If you discover $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ and are asked to find where the tangents to a curve are parallel to the y-axis (i.e. vertical) then you need to solve where the bottom line is zero, i.e. solve $g(x, y) = 0$.

Parametric Equations

- A parametric equation is one where
$$x = f(\text{some parameter}) \quad \text{and} \quad y = g(\text{some parameter}).$$
The parameter in a set of parametric equations can be any letter, but usually either t or θ. As the parameter varies it sketches out a curve. If no restriction is given, assume the parameter varies $-\infty < t < \infty$. However the parameter can be restricted in any way, defined by an inequality on the parameter. Standard examples: $0 \leqslant \theta < 2\pi$ or $-\pi < \theta \leqslant \pi$.
- You must be able to convert a parametric curve to Cartesian form. Sometimes this is just obvious; isolate t from one of the equations and put into the other. For example
$$\begin{array}{c} x = 2t \\ y = \frac{t}{t+1} \end{array} \quad \Rightarrow \quad t = \frac{x}{2} \quad \Rightarrow \quad y = \frac{\frac{x}{2}}{\frac{x}{2} + 1} = \frac{x}{x + 2}.$$
If one of x or y involves a "sin" and the other involves a "cos" then use $\sin^2 x + \cos^2 x = 1$:
$$\begin{array}{c} x = 3\cos\theta \\ y = \sin\theta + 4 \end{array} \quad \Rightarrow \quad \begin{array}{c} \cos^2\theta = \left(\frac{x}{3}\right)^2 \\ \sin^2\theta = (y - 4)^2 \end{array} \quad \Rightarrow \quad \frac{x^2}{9} + (y - 4)^2 = 1.$$

- To find where a line intersects a parametric curve, place the parameters (in terms of t) into the line and solve for t. For example find the points of intersection of

$$x = 2t^2 + 1, \quad y = \frac{1}{t} \qquad \text{and the line} \qquad x + 4y = 7.$$

Replace the x and y in the line by $2t^2 + 1$ and $\frac{1}{t}$ respectively. Therefore

$$x + 4y = 7, \qquad \Rightarrow \qquad (2t^2 + 1) + 4\left(\frac{1}{t}\right) = 7, \qquad \Rightarrow \qquad t^3 - 3t + 2 = 0.$$

This cubic factorises to $(t - 1)^2(t + 2) = 0$ which gives $t = 1$ or $t = -2$ as solutions. Plugging these back into the original parametric equation we discover the two points $(3, 1)$ and $(9, -\frac{1}{2})$. [It is worth noting that the squared factor $(t - 1)^2$ in the cubic implies the the line is a *tangent* to the curve at the point $(3, 1)$.]

- To differentiate a parametric curve

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

For stationary points you still equate $\frac{dy}{dx} = 0$ and solve. All other properties you are used to for normals and tangents still work.

For example find the equation of the normal to $x = 2t^3$, $y = \frac{1}{t}$ at the point $(16, \frac{1}{2})$. Firstly we need to discover the value of the parameter at the stated point: $y = \frac{1}{t} = \frac{1}{2}$ implies $t = 2$. Next differentiate and put in $t = 2$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-t^{-2}}{6t^2} = -\frac{1}{6t^4}.$$

$$\text{When } t = 2, \ \frac{dy}{dx} = -\frac{1}{96}.$$

Therefore the gradient of the normal is 96. Thus, $y - \frac{1}{2} = 96(x - 16)$ which 'simplifies' to $192x - 2y - 3071 = 0$.

- In harder examples questions will leave the parameter unevaluated; either leaving it as t or setting $t = p$. For example, find the equation of the tangent to the curve $x = 2t$, $y = \frac{1}{t^2}$ where $t = p$. When $t = p$, the point becomes $(2p, \frac{1}{p^2})$. Differentiating we find

$$\frac{dy}{dx} = \frac{-2t^{-3}}{2} = -\frac{1}{t^3}.$$

Therefore the gradient of the tangent when $t = p$ is $-\frac{1}{p^3}$. Therefore the tangent is

$$y - \frac{1}{p^2} = -\frac{1}{p^3}(x - 2p) \qquad \Rightarrow \qquad x + p^3 y = 3p.$$

The question could further be extended to find the area of the triangle formed by the points where the tangent crosses the x-axis and y-axis and the origin. The tangent $(x + p^3 y = 3p)$ crosses the x-axis when $y = 0$ which gives $x = 3p$. The tangent crosses the y-axis when $x = 0$ which gives $y = \frac{3}{p^2}$. So the three vertices of the triangle are at $(0, 0)$, $(0, \frac{3}{p^2})$ and $(3p, 0)$. The area of the triangle is therefore

$$\text{Area} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2} \times 3p \times \frac{3}{p^2} = \frac{3}{2p}.$$

Differential Equations

- If you are told that ("something") is proportional to ("something else") then we write ("something") $\propto$ ("something else"). This implies that

$$(\text{“something”}) = \pm k(\text{“something else”})$$

for some *constant* k. k can then be determined by putting in one pair of values (x, y) into the equation. If you read that something is decreasing then use $-k$, if it is increasing then use $+k$. The expression "varies as" also implies proportionality between two quantities.

- The words "rate of change of (something)" $\Rightarrow \frac{d(\text{something})}{dt}$. Also the word "initially" $\Rightarrow t = 0$. Also be on the lookout for phrases such as "where t is measured from now" implying $t = 0$ now.

- In simple cases you need to be able to construct a differential equation of a situation. For example: The number of people infected with bird flu (N) is growing at a rate proportional to the square of the number of people infected:

$$\frac{dN}{dt} \propto N^2 \quad \Rightarrow \quad \frac{dN}{dt} = +kN^2.$$

- The notation dy/dx lets us believe that it is a normal fraction. Although this is not the case we can manipulate it like a fraction in a differential equation. You must move the variables to different sides of the equation and integrate (separation of variables). Only add the ever-present "$+c$" to one side. For example solve

$$\frac{dy}{dx} = y^2\cos x \quad \Rightarrow \quad \int \frac{1}{y^2}\,dy = \int \cos x\,dx \quad \Rightarrow \quad y = -\frac{1}{\sin x + c}.$$
$$\frac{dN}{dt} = +kN^2 \quad \Rightarrow \quad \int \frac{1}{N^2}\,dN = \int k\,dt \quad \Rightarrow \quad N = \frac{-1}{kt + c}.$$

In the second example above you will notice that there are two constants; the constant of proportionality and the arbitrary integration constant. This means you will need to be given two pieces of data (t_1, N_1) and (t_2, N_2) to figure them both out.

- A final example involving partial fractions:

$$\begin{aligned}
(3P + 1)\frac{dP}{dt} &= kt(P - 1)(P + 3)\\
\int \tfrac{3P+1}{(P-1)(P+3)}\,dP &= \int kt\,dt\\
\int \tfrac{1}{P-1} + \tfrac{2}{P+3}\,dP &= \tfrac{kt^2}{2} + c\\
\ln(P - 1) + 2\ln(P + 3) &= \tfrac{kt^2}{2} + c\\
\ln(P - 1)(P + 3)^2 &= \tfrac{kt^2}{2} + c.
\end{aligned}$$

- If the arbitrary constant is left unevaluated, then your solution represents the *general solution* of the differential equation. If you put a value in to work out its value then your solution is called the *particular solution*.

Vectors

- The vector $\begin{pmatrix}3\\4\end{pmatrix}$ can be written $3\mathbf{i} + 4\mathbf{j}$ and represents a vector going 3 right and 4 up. By Pythagoras' Theorem it can be shown that the magnitude of this vector is $\sqrt{3^2+4^2} = 5$ and by trigonometry the direction is $\tan^{-1}\frac{4}{3}$ above the horizontal.
- Two vectors are equal if their magnitudes and directions are the same. Two vectors are parallel if one is a scalar multiple of the other. For example show that $\begin{pmatrix}2\\3\end{pmatrix}$ is parallel to $\begin{pmatrix}3\\4.5\end{pmatrix}$; so show that $1.5 \times \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}3\\4.5\end{pmatrix}$.
- When multiplying a vector by a positive scalar it changes the length of the vector but not the direction. If the scalar is negative then it also reverses the direction of the vector. For example $3\begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}6\\-9\end{pmatrix}$.
- When adding vectors, you just add the x components and add the y components. For example $\begin{pmatrix}2\\-3\end{pmatrix} + \begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$.
- A unit vector is a vector with a magnitude 1. A unit vector in a given direction can be constructed by dividing a vector by its magnitude. For example the unit vector from $\begin{pmatrix}2\\3\end{pmatrix}$ is $\frac{1}{\sqrt{2^2+3^2}} \times \begin{pmatrix}2\\3\end{pmatrix} = \frac{1}{\sqrt{13}} \times \begin{pmatrix}2\\3\end{pmatrix}$.
- You must know the geometric interpretation of $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$. Also know that in general if you have position vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.
- It cannot be stressed enough that *subtraction* is the most important operation with vectors. If you wish to travel *from* one point ($\mathbf{a}$) *to* another ($\mathbf{b}$) then we use subtraction: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

  I will repeat that! If you wish to travel *from* $\mathbf{a}$ *to* $\mathbf{b}$ then use subtraction: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.
- If you wish to calculate a length in 3D space then you merely need to calculate the magnitude of the vector that travels between the two points (i.e. $|\mathbf{b} - \mathbf{a}|$).
- A line can be written in vector form. If you know a line goes through a point (a, b) and has the gradient m then its vector form is $\mathbf{r} = \begin{pmatrix}a\\b\end{pmatrix} + \lambda\begin{pmatrix}1\\m\end{pmatrix}$ where λ is a scalar that takes different values on different points on the line. The vector $\begin{pmatrix}1\\m\end{pmatrix}$ can be re-written to make the components 'nicer'. For example $\begin{pmatrix}1\\2/3\end{pmatrix} \Rightarrow \begin{pmatrix}3\\2\end{pmatrix}$. (These vectors are not equal, but they have the same direction.) The most general form is

  $$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

  where $\mathbf{a}$ is the point it passes through and $\mathbf{d}$ is the direction vector (i.e. the vector that points *along* the line).

- We can therefore show that the equation of the line through **a** and **b** is given by $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, because $\mathbf{b} - \mathbf{a}$ is the vector that travels *from* **a** *to* **b** along the line. For example find the line that passes through $(2, 3, 1)$ and $(3, 6, -1)$. This gives

$$\mathbf{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\-2\end{pmatrix} \qquad \text{or} \qquad \mathbf{r} = \begin{pmatrix}3\\6\\-1\end{pmatrix} + \lambda\begin{pmatrix}-1\\-3\\2\end{pmatrix}.$$

- To find the angle between two vectors we use the scalar product result

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where $|\mathbf{u}|$ represents the magnitude of vector **u**. From this we can see that two vectors are perpendicular if their scalar product is zero.

- The scalar product is most easily calculated as follows; $\begin{pmatrix}a_x\\a_y\end{pmatrix} \cdot \begin{pmatrix}b_x\\b_y\end{pmatrix} = a_xb_x + a_yb_y$. (It is just a number, *not* a vector!)

- The following table sums up the 3D equivalents of the 2D results we have already found:

2D	3D
$\mathbf{i}, \mathbf{j}$	$\mathbf{i}, \mathbf{j}, \mathbf{k}$
$\lvert\mathbf{a}\rvert = \sqrt{a_x^2 + a_y^2}$	$\lvert\mathbf{a}\rvert = \sqrt{a_x^2 + a_y^2 + a_z^2}$
$\mathbf{a} \cdot \mathbf{b} = a_xb_x + a_yb_y$	$\mathbf{a} \cdot \mathbf{b} = a_xb_x + a_yb_y + a_zb_z$

Most of the results from the 2D section (above) still hold true for 3D vectors.

- Obviously in 2D provided lines have different gradient then they *must* intercept somewhere. However in 3D it is possible for two lines to have different direction vectors (i.e. not be parallel) and still not cross: these lines are called *skew*. This example shows how to discover if lines in 3D intercept or are skew.

$$\mathbf{r} = \begin{pmatrix}4\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\4\end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix}0\\-6\\5\end{pmatrix} + \mu\begin{pmatrix}2\\1\\1\end{pmatrix}.$$

Firstly we note the different direction vectors, so they cannot be parallel. Equate the x and y components[12] of both lines and solve for λ and μ:

$$\begin{aligned}(x):& \qquad 4 + \lambda = 2\mu,\\ (y):& \qquad -1 - \lambda = -6 + \mu.\end{aligned}$$

These solve to $\lambda = 2$ and $\mu = 3$. Put these values back into the original lines and compare z-coordinates: if they are the same then they intercept, if different then skew.

$$\mathbf{r} = \begin{pmatrix}4\\-1\\0\end{pmatrix} + 2\begin{pmatrix}1\\-1\\4\end{pmatrix} = \begin{pmatrix}6\\-3\\8\end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix}0\\-6\\5\end{pmatrix} + 3\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}6\\-3\\8\end{pmatrix}.$$

Therefore the lines cross at $(6, -3, 8)$. (You should find that the x and y-coordinates are *always* the same, it is only the z-coordinate that might be different; a nice little check!)

[12] You can take any pair of components you like here (x & y, x & z, or y & z) but most students just take x and y.

- To find the angle between two lines then *dot their direction vectors*. Using the two lines in the above example we find:

$$\begin{aligned}\begin{pmatrix}1\\-1\\4\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix} &= \sqrt{1^2+1^2+4^2}\sqrt{2^2+1^2+1^2}\cos\theta\\ 2-1+4 &= \sqrt{18}\sqrt{6}\cos\theta\\ \theta &= 61.2^\circ \text{ (to 3s.f.)}.\end{aligned}$$

  If you get an answer $90 < \theta \leqslant 180$ then give $180^\circ - \theta$ as your answer (Between any two lines there are two possible angles between them; think about it. The acute angle tends to be 'nicer').

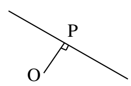
- When working out angles in 3D you must be very careful that you are 'dotting' the right vectors! For example if $A = (1, 2, -2)$, $B = (3, 1, -4)$ and $C = (7, 5, 1)$ find the angle $A\hat{B}C$. Draw a sketch! We require the angle at B so we need to dot $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

  Now $\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{pmatrix}-2\\1\\2\end{pmatrix}$ and $\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix}4\\4\\5\end{pmatrix}$. Therefore dotting we find:

$$\begin{aligned}\begin{pmatrix}-2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}4\\4\\5\end{pmatrix} &= \sqrt{9}\sqrt{57}\cos\theta\\ -8+4+10 &= 3\sqrt{57}\cos\theta\\ \frac{2}{\sqrt{57}} &= \cos\theta \qquad\Rightarrow\qquad \theta = 74.6^\circ\text{(to 3s.f.)}.\end{aligned}$$

- Some tough problems involve the use of

  "two vectors are at perpendicular" $\Leftrightarrow$ "the dot product is zero".

  For example find the point (P) on the line l: $\mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ closest to the origin. Firstly draw a sketch of a line running some distance past an origin. 

At the point P the vector $\overrightarrow{OP}$ must be perpendicular to the line. The direction vector is the vector *along* the line l, so we need

$$(\overrightarrow{OP}).\begin{pmatrix}1\\-1\\2\end{pmatrix} = 0.$$

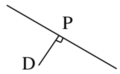
  The point P is some point on the line, so $P = \begin{pmatrix}1+\lambda\\2-\lambda\\2\lambda\end{pmatrix}$ for some λ. So $\overrightarrow{OP} = \begin{pmatrix}1+\lambda\\2-\lambda\\2\lambda\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1+\lambda\\2-\lambda\\2\lambda\end{pmatrix}$. Therefore

$$\begin{pmatrix}1+\lambda\\2-\lambda\\2\lambda\end{pmatrix}.\begin{pmatrix}1\\-1\\2\end{pmatrix} = 0.$$

This gives $1 + \lambda - 2 + \lambda + 4\lambda = 0$ which solves to $\lambda = \frac{1}{6}$. Putting this λ back into l we find

$$P = (\tfrac{7}{6}, \tfrac{11}{6}, \tfrac{1}{3}).$$

- Another tough example done in two ways: Find the shortest distance from point $D = (2, -1, 3)$ to the line $\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$.

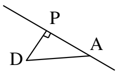
  - **Method I**: First method similar to above. Draw a sketch! 

 Let the point on the line closest to D be P. So $P = \begin{pmatrix}1+2\lambda\\-\lambda\\1+\lambda\end{pmatrix}$. We require $\overrightarrow{DP}$ to be perpendicular to the line if it is the closest point. Thus

$$\overrightarrow{DP}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix} = 0 \quad\Rightarrow\quad \left(\begin{pmatrix}1+2\lambda\\-\lambda\\1+\lambda\end{pmatrix} - \begin{pmatrix}2\\-1\\3\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix} = 0.$$

    This solves to $\lambda = \frac{5}{6}$. Therefore $P = (\frac{8}{3}, -\frac{5}{6}, \frac{11}{6})$. Therefore the distance is

$$\text{dist.} = |\overrightarrow{DP}| = |\mathbf{p} - \mathbf{d}| = \sqrt{(2 - \tfrac{8}{3})^2 + (\tfrac{5}{6} - 1)^2 + (3 - \tfrac{11}{6})^2} = \frac{\sqrt{66}}{6}.$$

  - **Method II**: Again, draw a sketch. 

 Let P be the point closest to D. This time also include the point that we know the line passes through $A = (1, 0, 1)$. We have therefore created a right angled triangle APD with a right angle at P. Length AD is just the magnitude of $\mathbf{d} - \mathbf{a}$; $|\mathbf{d} - \mathbf{a}| = \sqrt{(2-1)^2 + (-1-0)^2 + (3-1)^2} = \sqrt{6}$.
    Angle $D\hat{A}P$ can be worked out by $\overrightarrow{AD}\cdot$ (direction vector). So

$$(\mathbf{d} - \mathbf{a})\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix} = |\mathbf{d} - \mathbf{a}|\left|\begin{pmatrix}2\\-1\\1\end{pmatrix}\right|\cos\theta$$

$$\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix} = \sqrt{6}\sqrt{6}\cos D\hat{A}P.$$

    So $\cos D\hat{A}P = \frac{5}{6}$.
    By right angled trigonometry $\sin D\hat{A}P = \frac{DP}{\sqrt{6}}$. To convert a sin into a cos we use $\sin^2\theta + \cos^2\theta = 1$ which gives $\sin D\hat{A}P = \frac{\sqrt{11}}{6}$. Therefore $DP = \sqrt{6}\times\frac{\sqrt{11}}{6} = \frac{\sqrt{66}}{6}$, just as before.