OCR Further Pure Core Revision Sheet

The OCR further maths A level is examined with four 90 minute exams. Each paper carries equal weight (25%) and each paper is marked out of 75 marks. Two of the papers are compulsory:
Pure Core 1.
Pure Core 2.

Then you (or your school) selects two out of:
Statistics.
Mechanics.
Discrete Mathematics.
Additional Pure Mathematics.

This revision sheet *should* cover all of the pure core maths you need for the two compulsory papers. It represents 50% of your further maths A level. *Please* get in contact if you spot anything missing.

I hope you find this revision sheet useful and wish you the very best of luck with your studies.

J.M.S.

Summing Series

- You must know the important results:

$$\sum_{r=1}^{n} 1 = 1 + 1 + 1 + \cdots + 1 = n,$$
$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1),$$
$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1),$$
$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

- Also know the important properties (for constant λ and μ)

$$\sum_{r=1}^{n}(\lambda f(r) \pm \mu g(r)) = \lambda \sum_{r=1}^{n} f(r) \pm \mu \sum_{r=1}^{n} g(r).$$

However beware of these!

$$\sum_{r=1}^{n}(f(r) \times g(r)) \neq \sum_{r=1}^{n} f(r) \times \sum_{r=1}^{n} g(r) \qquad \text{and} \qquad \sum_{r=1}^{n}\left(\frac{f(r)}{g(r)}\right) \neq \frac{\sum_{r=1}^{n} f(r)}{\sum_{r=1}^{n} g(r)}.$$

To apply the first of these is equivalent to the heinous crime of $(a+b+c)^2 = a^2 + b^2 + c^2$!!!

- These must be applied in cases such as:

$$\begin{aligned}\sum_{r=1}^{n}(4r^2-2r+3) &= 4\sum_{r=1}^{n}r^2-2\sum_{r=1}^{n}r+3\sum_{r=1}^{n}1\\ &= \tfrac{2}{3}n(n+1)(2n+1)-n(n+1)+3n\\ &= \tfrac{n}{3}[2(n+1)(2n+1)-3(n+1)+9]\\ &= \tfrac{n}{3}(4n^2+3n+8).\end{aligned}$$

- If the sum starts from a number other than 1 then you can use the trick (which should be obvious)

$$\sum_{r=a}^{n}(\text{something}) = \sum_{r=1}^{n}(\text{something}) - \sum_{r=1}^{a-1}(\text{something}).$$

- The *method of differences* can be used to sum certain expressions where cancellation occurs when the sum is written out. For example find $\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$. Write the sum out, starting a new line for each value of r and you should see that some nice cancelling occurs;

$$\begin{aligned}\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right) &= \frac{1}{3}-\frac{1}{5}\\ &+\frac{1}{5}-\frac{1}{7}\\ &+\frac{1}{7}-\frac{1}{9}\\ &\vdots\\ &+\frac{1}{2n+1}-\frac{1}{2n+3}.\end{aligned}$$

  You can see that everything cancels except the $\frac{1}{3}$ and the $\frac{1}{2n+3}$ so

$$\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right) = \frac{1}{3}-\frac{1}{2n+3}.$$

  It is usually best *not* to combine these terms together into one fraction in order to make it easier to see if there is a sum to infinity.

- A sum to infinity exists if the expression for the sum to n has a finite limit as $n \to \infty$. In the above example it does, so

$$\sum_{r=1}^{\infty}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right) = \lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2n+3}\right) = \frac{1}{3}.$$

- Questions of this sort invariably start "Show that $f(r)-g(r)=h(r)$", and then ask you to sum $h(r)$; this, clearly, is the same as summing $f(r)-g(r) \Rightarrow$ 'method of differences'.

Matrices

- Capital letters tend to be used to denote matrices and you should underline them, just as you do with vectors. An $n \times m$ matrix has n rows and m columns. So $\begin{pmatrix} 1 & 2 & -3 \\ 2 & -2 & 7 \end{pmatrix}$ is a 2×3

matrix. You must be able to add, subtract and multiply matrices. To add or subtract matrices they must be the same size and it works as you would expect. To multiply matrices ($\mathbf{A} \times \mathbf{B}$, say) the number of columns of $\mathbf{A}$ must be the same as the number of rows of $\mathbf{B}$. Your teacher will have explained this better than I ever can here, but a few examples: test for yourself!

$$\begin{pmatrix} 1 & 5 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 7 & -1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 12 & 19 \\ -4 & -13 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}\begin{pmatrix} 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 0 & 0 \\ 8 & 12 & -4 \end{pmatrix}.$$

- The determinant of an $n \times n$ ('*square*') matrix can be denoted by the letter Δ. A matrix with $\Delta = 0$ is called a '*singular*' matrix; otherwise it is '*non-singular*'. For a 2×2 matrix $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \Delta = ad - bc$.
- The inverse of a 2×2 matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- The inverse of a matrix (if it exists) is such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- Matrix multiplication is not, in general, commutative; i.e. $\mathbf{AB} \neq \mathbf{BA}$. Matrix multiplication is, however, associative; i.e. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. An extension of this is $(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 \ldots \mathbf{A}_n)^{-1} = \mathbf{A}_n^{-1} \ldots \mathbf{A}_3^{-1}\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$; prove it by induction yourself if you fancy…
- You must be very careful when manipulating matrix equations because of this non-commutativity. With normal numbers we are happy with $ax = b$ giving $x = ba^{-1}$, but this is wrong in matrix-world.

$$\begin{aligned} \mathbf{AX} &= \mathbf{B} \\ \mathbf{A}^{-1}\mathbf{AX} &= \mathbf{A}^{-1}\mathbf{B} \quad \underline{\text{pre}}\text{-multiply both sides by } \mathbf{A}^{-1} \\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B}. \end{aligned}$$

Or

$$\begin{aligned} \mathbf{XA} &= \mathbf{B} \\ \mathbf{XAA}^{-1} &= \mathbf{BA}^{-1} \quad \underline{\text{post}}\text{-multiply both sides by } \mathbf{A}^{-1} \\ \mathbf{XI} &= \mathbf{BA}^{-1} \\ \mathbf{X} &= \mathbf{BA}^{-1}. \end{aligned}$$

- Know that linear simultaneous equations can be expressed by matrices:

$$\begin{matrix} ax + by = c \\ dx + ey = f \end{matrix} \quad \Rightarrow \quad \begin{pmatrix} a & b \\ d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}.$$

The system of equations has a unique solution provided $\Delta \neq 0$. If $\Delta = 0$ then there are no unique solutions: there are either an infinite set of solutions or no solutions at all depending on whether $ax + by = c$ and $dx + ey = f$ represent parallel lines (no solutions) or the same line (infinite set of solutions).

The unique solution (if it exists) is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ae - bd}\begin{pmatrix} e & -b \\ -d & a \end{pmatrix}\begin{pmatrix} c \\ f \end{pmatrix}$.

Matrix Transformations

- Matrices can be thought of as transformations. To discover what a matrix does, consider what it does to the arbitrary point (x, y) and *think*! For example

  1. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ so $(x, y) \to (x, y)$. Therefore matrix does nothing.
  2. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ so $(x, y) \to (y, x)$. Therefore the x and y-coordinates get flipped, so matrix reflected in the line $y = x$.
  3. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$ so $(x, y) \to (-x, y)$. Therefore matrix changes the sign of the x-coordinate, so it represents a reflection in the y-axis.
  4. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$ so $(x, y) \to (y, -x)$. Draw a few sample points and we see it represents a rotation 90° clockwise about the origin.
  5. $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$ so $(x, y) \to (4x, 4y)$. So the x and y-coordinates get multiplied by 4. Therefore an enlargement scale factor 4, centre the origin.

- You need to know the family of matrices that represent shears.

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \text{“Shear with } x\text{-axis invariant with shear constant } k\text{”.} \quad \rightleftarrows$$

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} = \text{“Shear with } y\text{-axis invariant with shear constant } k\text{”.} \quad \downarrow\uparrow$$

- For combined transformations you write the matrices in the opposite order to which the transformations occur[1]. For example if we apply transformation **A** followed by transformation **B**, then the matrix for this combined transformation would be **BA**.

- If a 2×2 matrix $\mathbf{M}$ represents a transformation, then $|\det(\mathbf{M})|$ represents the *area scale factor* of the transformation.

  For example $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents an enlargement with length scale factor 2. We see the determinant is 4, so areas get multiplied by 4 in the transformation, which is consistent.

- If we consider an arbitrary matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on the point $(1, 0)$ we find

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

[1] Just like functions: If you apply f then g, we do $gf(x)$.

(i.e. the first column of the matrix). Similarly if we act on the point $(0, 1)$ we find

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

(i.e. the second column of the matrix). This immensely powerful pair of statements tells us that *if* a transformation can be expressed by a matrix, then all we need to do to find the matrix that does what we want is to find where $(1, 0)$ maps to under the transformation and write this image point as the first column of our matrix and find where $(0, 1)$ maps to under the transformation and write this as the second column.

- For example find the matrix that:
  - reflects in the x–axis. $(1, 0) \to (1, 0)$ and $(0, 1) \to (0, -1)$ $\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
  - reflects in the $y = x$. $(1, 0) \to (0, 1)$ and $(0, 1) \to (1, 0)$ $\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
  - rotates 90° clockwise. $(1, 0) \to (0, -1)$ and $(0, 1) \to (1, 0)$ $\Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
  - enlarges scale factor 3. $(1, 0) \to (3, 0)$ and $(0, 1) \to (0, 3)$ $\Rightarrow \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.
  - stretch factor 2 parallel to y–axis. $(1, 0) \to (1, 0)$ and $(0, 1) \to (0, 2)$ $\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.
  - rotates θ° anticlockwise. $(1, 0) \to (\cos\theta, \sin\theta)$ and $(0, 1) \to (-\sin\theta, \cos\theta)$ $\Rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
  - rotates 90° CW and then reflects in y–axis. $(1, 0) \to (0, -1)$ and $(0, 1) \to (1, 0)$ $\Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3×3 Matrices

- To calculate the determinant of a 3×3 matrix, you pick a column or a row (most students choose the first column, but it works with any row or column) and you work down/across it using the plus/minus checkerboard approach and multiplying by the determinant of the 2×2 matrix left when the column and row of the number you have chosen is crossed out.

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \Delta = a(ei - fh) - d(bi - ch) + g(bf - ce).$$

- To invert a 3×3 matrix you do:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} +(ei - hf) & -(di - gf) & +(dh - eg) \\ -(bi - ch) & +(ai - cg) & -(ah - bg) \\ +(bf - ce) & -(af - cd) & +(ae - bd) \end{pmatrix}^{\mathrm{T}},$$

$$= \frac{1}{\Delta}\begin{pmatrix} +(ei - hf) & -(bi - ch) & +(bf - ce) \\ -(di - gf) & +(ai - cg) & -(af - cd) \\ +(dh - eg) & -(ah - bg) & +(ae - bd) \end{pmatrix}.$$

Don't forget to transpose at the end! There is an elegant pattern to all of the above; it's easy to do once you get into the swing of it. (Mr Stone has a spreadsheet where you can practice this to your heart's content.)

- As before, a system of linear simultaneous equations can be written with a matrix.

$$\begin{array}{r} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kz=l \end{array} \quad \Rightarrow \quad \begin{pmatrix} a & b & c \\ e & f & g \\ i & j & k \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} d \\ h \\ l \end{pmatrix}.$$

  If the matrix is non-singular then the equations have a unique solution. If the matrix is singular then the system either has an infinite set of solutions or no solutions at all. If the equations generate an inconsistency (e.g. 4=19) then there are no solutions at all.

- If the matrix is non-singular, then the unique solution is given by:

$$\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Complex Numbers

- Complex numbers start with one idea only; that we can find a number that squares to -1; we call it i. Therefore $i^2 = -1$. It is not a number that exists on the number line so it is refered to as *complex* or *imaginary*. Therefore the square root of any negative number can now be calculated; $\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6i$.

- In general a complex number can consist of a real part and an imaginary/complex part i.e. $a+ib$, where a is the real part and b is the complex part. We write $\mathrm{Re}(a+ib) = a$ and $\mathrm{Im}(a+ib) = b$. It is important to note that a and b themselves *must* be real numbers.

- We can use complex numbers to solve *any* quadratic equation. For example solve $3x^2+2x+7 = 0$ by the formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{2^2-4\times 3\times 7}}{2\times 3} = -\frac{1}{3} \pm i\frac{2\sqrt{5}}{3}.$$

- A complex (or real) number can be represented as a point in an *Argand* diagram. So the complex number $6+2i$ would be the point 6 across and 2 up, at the equivalent point where $(6,2)$ would be in a Cartesian coordinate system.

- The complex conjugate ($z*$) of a complex number (z) is where the complex part has the sign changed. For example if $z = 3-7i$, then $z* = 3+7i$. Real numbers are, therefore, their own conjugates. Any number with an imaginary component is reflected in the real axis in the Argand diagram.

- If two complex numbers are equal, then the real parts must be equal and the complex parts must be equal: i.e.

$$\begin{array}{rcl} z_1 = z_2 & \Rightarrow & \mathrm{Re}(z_1) = \mathrm{Re}(z_2) \quad \text{and} \quad \mathrm{Im}(z_1) = \mathrm{Im}(z_2), \\ a+ib = c+id & \Rightarrow & a = c \quad \text{and} \quad b = d. \end{array}$$

- To add, subtract or multiply complex numbers the results are pretty obvious:

$$\begin{aligned} (a+bi)+(c+id) &= (a+c)+i(b+d) \\ (a+bi)-(c+id) &= (a-c)+i(b-d) \\ (a+bi)(c+id) &= ac+adi+bci+bdi^2 = (ac-bd)+i(ad+bc) \end{aligned}$$

- To divide by a complex number use a trick taken from surds; in C1 if the bottom line was $a \pm b\sqrt{k}$ then you multiplied top and bottom by $a \mp b\sqrt{k}$. In FP1 if you want to divide by $a \pm ib$, then you multiply top and bottom by the complex conjugate $a \mp ib$. For example:

$$\frac{3-2i}{2+5i} = \frac{3-2i}{2+5i} \times \frac{2-5i}{2-5i} = \frac{6+10i^2-4i-15i}{4-25i^2+10i-10i} = \frac{-4-19i}{29}.$$

- Complex numbers exhibit the elegant property of *closure*[2]. This means that any operation on complex numbers involving $+, -, \times, \div, \sqrt{\ldots}, \sqrt[n]{\ldots}$ etc. will produce an answer that is also complex[3]. This allows us to state that the answer to a given problem *must* be $a + ib$ for some a and b and then proceed to calculate a and b by equating the real part and, separately, the imaginary part.

- In the above problem to find $\frac{3-2i}{2+5i}$ we could also have approached it by stating that the answer is $a + ib$ and manipulating:

$$\begin{aligned} \frac{3-2i}{2+5i} &= a + ib \\ 3 - 2i &= (a + ib)(2 + 5i) \\ 3 - 2i &= (2a - 5b) + i(5a + 2b). \end{aligned}$$

  This yields the simultaneous equations $3 = 2a - 5b$ and $-2 = 5a + 2b$. These solve to $a = -\frac{4}{29}$ and $b = -\frac{19}{29} \Rightarrow \frac{-4-19i}{29}$, just as before. I wouldn't use this method in this case but I would certainly use it…

- …to find square roots. The square roots of 16 are (obviously) ± 4. With the exception of zero, we should expect two roots and the same is true of complex numbers. For example find the square roots of $8 - 6i$: We know that the answers must be of the form $a + ib$ such that

$$\begin{aligned} 8 - 6i &= (a + ib)^2 \\ 8 - 6i &= (a^2 - b^2) + (2ab)i \\ \text{Therefore, } 8 &= a^2 - b^2 \text{ and } -6 = 2ab. \end{aligned}$$

  From the second we find $b = -\frac{3}{a}$. Putting this in the first we find $0 = a^4 - 8a^2 - 9 = (a^2 - 9)(a^2 + 1)$. The first bracket yields $a = \pm 3$. (The second bracket yields $a = \pm i$, but we can discard this because a must be real.) Therefore this yields the square roots $3 - i$ and $-3 + i$. In the Argand diagram you should find that square roots come out in opposite directions from the origin.

- If a polynomial has *real coefficients* then its roots are either real, or exist in complex conjugate pairs. Therefore if $z = a \pm ib$ is a root, then so is $z = a \mp ib$.

  For example, given that $z^4 - z^3 + 2z^2 + 7z - 5 = 0$ has one root $1 - 2i$, solve the equation fully. Since the coefficients are real we know that the conjugate $1 + 2i$ must also be a root. Therefore $(z - (1 - 2i))$ and $(z - (1 + 2i))$ must be factors by the factor theorem. Multiplying out the two factors we find $(z - (1 - 2i))(z - (1 + 2i)) = (z^2 - 2z + 5)$ which must also be a factor. By polynomial division we find

$$z^4 - z^3 + 2z^2 + 7z - 5 = (z^2 - 2z + 5)(z^2 + z - 1) = 0.$$

  The second quadratic solves to $z = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Therefore the solutions are

$$z = 1 - 2i, \quad z = 1 + 2i, \quad z = -\tfrac{1}{2} + \tfrac{\sqrt{5}}{2}, \quad z = -\tfrac{1}{2} - \tfrac{\sqrt{5}}{2}.$$

[2]"And that, my friend, is what they call closure" - *Rachel Green, Friends*

[3]Note this does not happen with the real numbers; you cannot always square root a number.

- The modulus of a complex number ($z = x + iy$) is defined $|z| = \sqrt{x^2 + y^2}$. It represents the distance of a complex number from the origin. For example the modulus of $z = 2 - 2\sqrt{3}i$ would be $|z| = \sqrt{2^2 + (-2\sqrt{3})^2} = 4$.

- The argument of a complex number is defined as the angle a line from the origin to a complex number makes with the positive real axis. By convention $-\pi < \arg(z) \leqslant \pi$. For example

$$\begin{aligned}\arg(4) &= 0,\\ \arg(i) &= \frac{\pi}{2},\\ \arg(-3) &= \pi,\\ \arg(1+i) &= \frac{\pi}{4},\\ \arg(-1-i) &= -\frac{3\pi}{4}.\end{aligned}$$

  Arguments are best calculated by drawing a suitable right angled triangle in an Argand diagram and then calculating the desired angle (not always an angle in the triangle you've drawn, but π minus it, etc.)

- You must be able to sketch loci of points obeying a rule defined by a modulus or an argument. The most important fact here is often that the operation *subtraction* takes you *from* one complex number *to* another[4]; i.e. $z - w$ takes you *from* w *to* z. An addition can be converted into a subtraction by $z + w = z - (-w)$; this therefore represents the movement from $-w$ to z.

- This idea allows us to draw certain loci very easily indeed.

  For example: $|z| = 4$ means the length of z from the origin is 4; i.e. a circle of radius 4, centre the origin.

  For example: $|z-2| = 5$ means the length travelling from 2 to z is 5, so a circle radius 5, centre 2(+0i).

  For example: $|z + i| < 2$ is the same as $|z - (-i)| < 2$ which means the length travelling from $-i$ to z is less than 2, so the inside of a circle radius 2, centre $(0) - i$.

  For example: $|z| = |z + 1 - i|$ is the same as $|z| = |z - (-1 + i)|$ which means the length travelling from 0 to z must be the same as the distance travelling from $-1 + i$ to z so it must be the perpendicular bisector of 0 and $-1 + i$, i.e. $y = x + 1$.

- The above type of questions can also be done using a method in your textbook (see top of P144, "Method 2"), but I prefer the 'intuitive' way demonstrated above.

- Argument loci also come up and we can use the same principles.

  For example: $\arg(z) = \frac{\pi}{2}$ means the argument z makes is $\frac{\pi}{2}$ so it is a vertical line going up from the origin (with a hollow circle drawn at the origin to indicate that it is not included in the half-line).

  For example: $\arg(z-i) = \frac{\pi}{6}$ means the argument going from i to z is $\frac{\pi}{6}$ so it is a half-line from i (hollow circle) at angle $\frac{\pi}{6}$ with the positive real axis.

[4] In precisely the same way that with vectors $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Roots Of Equations

- By considering the general quadratic equation $ax^2+bx+c=0$ we re-write it as $x^2+\frac{b}{a}x+\frac{c}{a}=0$. Quadratics can be factorised into two linear factors $(x-\alpha)(x-\beta)$. By equating the two we find

$$(x-\alpha)(x-\beta) = x^2+\frac{b}{a}x+\frac{c}{a}$$
$$x^2-(\alpha+\beta)x+\alpha\beta = x^2+\frac{b}{a}x+\frac{c}{a}.$$

  So we see that the sum of the roots of a quadratic is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$.

- By two tedious derivations (that you *should* do for yourself) similar to the one above we find that for the cubic $(ax^3+bx^2+cx+d=0)$ and the quartic $(ax^4+bx^3+cx^2+dx+e=0)$ the following:

Quadratics	Cubics	Quartics
$\alpha+\beta=-\frac{b}{a}$	$\alpha+\beta+\gamma=-\frac{b}{a}$	$\alpha+\beta+\gamma+\delta=-\frac{b}{a}$
$\alpha\beta=\frac{c}{a}$,	$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$	$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}$
	$\alpha\beta\gamma=-\frac{d}{a}$,	$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}$
		$\alpha\beta\gamma\delta=\frac{e}{a}$.

  For your exam you only need quadratics and cubics, but the pattern continues fairly easily to quartics, quintics and beyond.

- For speed of writing we use the following shorthand:

$$\alpha+\beta+\gamma\equiv\sum\alpha \qquad \text{and} \qquad \alpha\beta+\alpha\gamma+\beta\gamma\equiv\sum\alpha\beta.$$

- We can therefore find properties of roots from equations without having to solve the equations themselves.

  For example from $2x^2-3x-6=0$ I can say that $\alpha\beta=\frac{-6}{2}=-3$ and $\alpha+\beta=\frac{3}{2}$.

  For example from $2x^3-4x^2-3x+6=0$ I can say that $\alpha\beta\gamma=\frac{-6}{2}=-3$, $\sum\alpha\beta=-\frac{3}{2}$ and $\sum\alpha=\frac{4}{2}=2$. Watch those signs!

- You are often asked to construct new equations with roots related to the original equation's roots. There are two basic methods for this:

- Method I (the "Lo-Tech" approach) is to find the new 'sum'/'sum of prod'/'prod' of roots etc., from knowledge of the old 'sum'/'sum of prod'/'prod'. Two examples:

  1. The equation $2x^2+5x+7=0$ has roots α and β. Find an equation with roots $2\alpha-1$ and $2\beta-1$. We can see that $\alpha\beta=\frac{7}{2}$ and $\alpha+\beta=-\frac{5}{2}$. Therefore for the new equation must have the following:
     New sum of roots $=(2\alpha-1)+(2\beta-1)=2(\alpha+\beta)-2=2\times(-\frac{5}{2})-2=-7$.
     New prod of roots $=(2\alpha-1)(2\beta-1)=4\alpha\beta-2(\alpha+\beta)+1=4\times\frac{7}{2}-2\times(-\frac{5}{2})+1=20$.
     Therefore the new equation is $u^2+7u+20=0$.

2. The equation $2x^3 - x^2 + 4x + 2 = 0$ has roots α, β and γ. Find an equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$. We can see that $\alpha\beta\gamma = -1$, $\sum \alpha\beta = 2$ and $\sum \alpha = \frac{1}{2}$. Therefore for the new equation we must have the following:
   New sum of roots $= (\alpha + 1) + (\beta + 1) + (\gamma + 1) = \sum \alpha + 3 = \frac{7}{2}$.
   New sum of prods $= (\alpha+1)(\beta+1)+(\alpha+1)(\gamma+1)+(\beta+1)(\gamma+1) = \sum \alpha\beta + 2\sum \alpha + 3 = 2 + 2 \times \frac{1}{2} + 3 = 6$.
   New prod of roots $= (\alpha+1)(\beta+1)(\gamma+1) = \alpha\beta\gamma + \sum \alpha\beta + \sum \alpha + 1 = -1 + 2 + \frac{1}{2} + 1 = \frac{5}{2}$.
   Therefore the equations becomes $u^3 - \frac{7}{2}u^2 + 6u - \frac{5}{2} = 0$ which we double to make it 'nice':

$$2u^3 - 7u^2 + 12u - 5 = 0.$$

- Method II (the "Hi-Tech" approach) is to make a substitution into the original equation to construct a second. We will do the same two examples as above.

1. The equation $2x^2 + 5x + 7 = 0$ has roots α and β. Find an equation with roots $2\alpha - 1$ and $2\beta - 1$. Let u be one of the new roots; $u = 2\alpha - 1$. Rearrange to make α the subject; $\alpha = \frac{u+1}{2}$. We know that α satisfies the original equation because it is a root, so if we substitute $\alpha = \frac{u+1}{2}$ into the original equation we will have an equation in u which has the desired roots.

$$2\left(\frac{u+1}{2}\right)^2 + 5\left(\frac{u+1}{2}\right) + 7 = 0 \quad \Rightarrow \quad u^2 + 7u + 20 = 0.$$

2. The equation $2x^3 - x^2 + 4x + 2 = 0$ has roots α, β and γ. Find an equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$. So, let $u = \alpha + 1$. Therefore $\alpha = u - 1$. Sub in we find:

$$2(u-1)^3 - (u-1)^2 + 4(u-1) + 2 = 0 \quad \Rightarrow \quad 2u^3 - 7u^2 + 12u - 5 = 0.$$

Proof by Induction

- Let $P(n)$ be a proposition which depends on some integer value n. The principle of induction works as follows: Start by demonstrating the truth of $P(1)$ (say). Then we show that *if* $P(k)$ is true for *some value k then* it implies the truth of $P(k+1)$, then $P(n)$ must be true for all integer $n \geqslant 1$. This is because we have shown

$$P(1) \Rightarrow P(2), \text{ and } P(2) \Rightarrow P(3), \text{ and } P(3) \Rightarrow P(4) \text{ etc. etc. etc.}$$

- Your answer should always follow this template:
  - "Let $P(n)$ be the proposition that $f(n) = g(n)$ for all $n \geqslant 1$."
  - "Basis Case: If $n = 1$, $f(1) = \ldots$ and $g(1) = \ldots$. We see $f(1) = g(1)$ so $P(1)$ is true."
  - "Let us suppose that $P(n)$ is true for some $n = k$:

$$f(k) = g(k)."$$

    [Then manipulate $f(k) = g(k)$ using algebra to obtain the next line]

$$"f(k+1) = g(k+1)."$$

  - "This is the statement of $P(k+1)$."
  - "Therefore we have shown that *if* $P(k)$ is true *then* $P(k+1)$ is also true and since $P(1)$ is also true we can conclude by the principle of mathematical induction that $P(n)$ is true for all $n \geqslant 1$."

- If an induction question includes a "$\sum$", can I suggest you get rid of it by writing out the sum term-by-term; students tend to get muddled on when to use r, n, k and $k+1$ in my experience (although this might be my teaching). Also leave initial numerical values unevaluated; 1×2^2 is preferable to 4. For example

$$\sum_{r=1}^{n} r(r+2) \Rightarrow 1 \times 3 + 2 \times 4 + \cdots + n(n+2).$$

- For example use induction to prove that for $n \geqslant 2$, $\sum_{r=2}^{n}(r-1)r = \frac{1}{3}n(n-1)(n+1)$.
  - "The question is the same as proving $1 \times 2 + 2 \times 3 + \cdots + (n-1)n = \frac{1}{3}n(n-1)(n+1)$."
  - "Let $P(n)$ be the proposition $1 \times 2 + 2 \times 3 + \cdots + (n-1)n = \frac{1}{3}n(n-1)(n+1)$."
  - "Basis case: If $n = 2$, $1 \times 2 + 2 \times 3 + \cdots + (n-1)n = 2$ and $\frac{1}{3}n(n-1)(n+1) = 2$. We see that LHS = RHS = 2 so $P(2)$ is true."
  - "Let us suppose that $P(n)$ is true for some $n = k$:

$$1 \times 2 + 2 \times 3 + \cdots + (k-1)k = \frac{1}{3}k(k-1)(k+1)."$$

  - (Add the next term to the LHS to both sides:)

$$\begin{aligned} \text{“}1 \times 2 + 2 \times 3 + \cdots + (k-1)k + \underline{k(k+1)} &= \frac{1}{3}k(k-1)(k+1) + \underline{k(k+1)} \\ &= \frac{1}{3}k(k+1)\big[(k-1)+3\big] \\ &= \frac{1}{3}k(k+1)(k+2).\text{”} \end{aligned}$$

  - "This is the statement of $P(k+1)$."
  - "Therefore we have shown that **if** $P(k)$ is true **then** $P(k+1)$ is also true and since $P(2)$ is also true we can conclude by the principle of mathematical induction that $P(n)$ is true for all $n \geqslant 2$."

- In a recent official mark scheme, the use of the words 'mathematical induction' in your conclusion was needed for full marks.

Rational Functions

- Review all partial fractions and polynomial division work from C4 before starting this section.
- A nice 'trick' that can be used from time-to-time is to make the top line of an algebraic fraction look like a multiple of the bottom. Then you can split it up. For example

$$\frac{x-1}{x+3} = \frac{x+3-4}{x+3} = 1 - \frac{4}{x+3},$$
$$\frac{2x^2-1}{x-2} = \frac{2x^2-4x+4x-1}{x-2} = \frac{2x(x-2)+4x-1}{x-2} = 2x + \frac{4x-8+7}{x-2} = 2x+4+\frac{7}{x-2}.$$

  Some students like this & others don't; it's up to you if you use it. You could just use polynomial division.

- For some reason best known to the examiners at OCR, C4 only contains two of the three partial fraction types[5]. In C4 you dealt with $\frac{ax+b}{(cx+d)(ex+f)}$ and $\frac{ax^2+bx+c}{(dx+e)(fx+g)^2}$. In FP2 you also need to know how do deal with $\frac{ax^2+bx+c}{(dx+e)(fx^2+g)}$. The general technique is

$$\frac{ax^2+bx+c}{(dx+e)(fx^2+g)} \equiv \frac{A}{dx+e} + \frac{Bx+C}{fx^2+g}.$$

Remember that to use 'pure' partial fractions the numerator has to have order less than the denominator.

- For example to express $\frac{5x^2-7x+14}{(x-3)(2x^2+1)}$ in partial fractions we start:

$$\frac{5x^2-7x+14}{(x-3)(2x^2+1)} \equiv \frac{A}{x-3} + \frac{Bx+C}{2x^2+1},$$
$$\Rightarrow \qquad 5x^2-7x+14 \equiv (2x^2+1)A + (x-3)(Bx+C).$$

Clearly a good x-value to use is $x = 3$, so

$$x = 3 \qquad \Rightarrow \qquad 45 - 21 + 14 \equiv 19A \qquad \Rightarrow \qquad \underline{A = 2}.$$

We've no more cunning values so just use $x = 0$ to find $14 = 2 - 3C$, which gives $\underline{C = -4}$. Next use $x = 1$ to give $12 = 6 + (-2)(B - 4)$, which solves to $\underline{B = 1}$. Therefore

$$\frac{5x^2-7x+14}{(x-3)(2x^2+1)} \equiv \frac{2}{x-3} + \frac{x-4}{2x^2+1}.$$

- You can always make the leap from polynomial division to partial fractions in one go if you like. For example to divide

$$\frac{3x^4+13x^3+27x^2+56x+59}{x^3+3x^2+4x+12}$$

we have '$\frac{\text{quartic}}{\text{cubic}}$'. We are therefore expecting 'linear + $\frac{\text{quadratic}}{\text{cubic}}$'. But, because the denominator can be factorised to $(x+3)(x^2+4)$, we could split the '$\frac{\text{quadratic}}{\text{cubic}}$' term into partial fractions too. So

$$\frac{3x^4+13x^3+27x^2+56x+59}{(x+3)(x^2+4)} \equiv Ax + B + \frac{C}{x+3} + \frac{Dx+E}{x^2+4},$$
$$3x^4+13x^3+27x^2+56x+59 \equiv (Ax+B)(x+3)(x^2+4) + C(x^2+4) + (Dx+E)(x+3),$$

Clearly $A = 3$ by considering the x^4 coefficient. Running through the rest of the calculations in the usual way (you should do this yourself) we find

$$\frac{3x^4+13x^3+27x^2+56x+59}{(x+3)(x^2+4)} \equiv 3x + 4 + \frac{2}{x+3} + \frac{x+1}{x^2+4}.$$

[5] Note that the wonderful MEI has all three in C4 which is much more coherent...

Graphs

- To sketch a graph of $y = \frac{f(x)}{g(x)}$ there are a series of steps to follow. If one step contradicts another, chances are you've made a mistake. Firstly to find where a curve crosses the x-axis, set $y = 0$ and solve. Similarly to find where a curve crosses the y-axis, set $x = 0$ and solve.
- To find stationary points just solve $\frac{dy}{dx} = 0$ as usual. To discover their nature you can use the second derivative as normal; or the lo-tech approach. Review your C1 notes.
- To find the vertical asymptotes of the curve $y = \frac{f(x)}{g(x)}$ you need to find where $g(x) = 0$. So $y = \frac{x+3}{(2x-1)(x+2)}$ will have vertical asymptotes $x = -2$ and $x = \frac{1}{2}$.
- To find a horizontal asymptotes of $y = \frac{f(x)}{g(x)}$ you must look at the order of $f(x)$ and the order of $g(x)$.
  1. If "order of $f(x)$" $<$ "order of $g(x)$" then, as $x \to \pm\infty$, $g(x)$ is much, *much* larger than $f(x)$, so $y = 0$ is the horizontal asymptote.
  2. If "order of $f(x)$" $=$ "order of $g(x)$" then, as $x \to \pm\infty$, the dominant term of $f(x)$ and $g(x)$ becomes the largest power of x. Therefore the horizontal asymptote becomes the ratio of the coefficients of the largest power of x. For example $y = \frac{7x^2+2x-1}{5x^2-x-2}$ has $y = \frac{7}{5}$ as its horizontal asymptote.

     (Another, possibly better, way of thinking about this to divide the improper fraction into quotient and remainder by polynomial division. Since the order of the numerator is the same as the order of the denominator, then the quotient is a constant. This constant is the value of the horizontal asymptote.)
  3. If "order of $f(x)$" $>$ "order of $g(x)$" then there is *no* horizontal asymptote because as $x \to \pm\infty$, $f(x)$ is much, *much* larger than $g(x)$, so y is unbounded, heading up to $+\infty$ or down to $-\infty$ (just think about what happens when x gets really big).
- If the numerator has order one more than the denominator, there will exist an *oblique asymptote*; a line of the form $y = mx + c$ that the curve approaches when $x \to \pm\infty$. To find this line, you must carry out the polynomial division and find the quotient and remainder. For example $y = \frac{3x^2+4x+5}{x+1}$. We know $y = \frac{3x^2+4x+5}{x+1} = Ax + B + \frac{C}{x+1}$ and, carrying out the calculation (do it yourself!), we find $y = 3x + 1 + \frac{4}{x+1}$. Therefore the oblique asymptote is $y = 3x + 1$ because the $\frac{4}{x+1} \to 0$ as $x \to \pm\infty$.

  Similarly you should find (again, do it yourself!) $y = \frac{-x^3+x+2}{x^2+x+1} = -x+1+\frac{x+1}{x^2+x+1}$, so the oblique asymptote is $y = -x + 1$.
- You must be able to discover the range of y-values for which the curve exists, and, equivalently, the values for which it does not. This can be done by finding the stationary points on the curve and considering a sketch. However, there is quite a neat algebraic method. For example: Find the values of y for which the curve

  $$y = \frac{x^2 + x + 1}{x^2 + 1}$$

  exists. Multiplying by the denominator we discover $y(x^2 + 1) = x^2 + x + 1$. This can be rearranged as a quadratic in x: $(y - 1)x^2 - x + (y - 1) = 0$. For the curve to exist, we need the quadratic to have at least one solution, so $b^2 - 4ac \geqslant 0$. So

  $$\begin{aligned}(-1)^2 - 4(y-1)(y-1) &\geqslant 0,\\ 4y^2 - 8y + 3 &\leqslant 0,\\ (2y-3)(2y-1) &\leqslant 0.\end{aligned}$$

This quadratic inequality solves to $\frac{1}{2} \leqslant y \leqslant \frac{3}{2}$. So the curve only exists between the horizontal lines $y = \frac{1}{2}$ and $y = \frac{3}{2}$.

- Given a graph of $y = f(x)$, you must also be able to sketch the graph of $y^2 = f(x)$. Most students (including myself) mentally re-cast the problem as drawing $y = \pm\sqrt{f(x)}$. Things to look for include

  1. Anything below the x-axis on the original graph (y) is negative and therefore *cannot* be square rooted. Therefore these x-values represent a forbidden region where y^2 doesn't exist.
  2. All y-values above the x-axis get square rooted. Then these new points also get reflected in the x-axis. *Any* graph $y^2 = f(x)$, *must* have a line of symmetry in the x-axis.
  3. All positive y-values on the original graph get square rooted; therefore points on the line $y = 1$ are invariant. If $y > 1$, then they get 'pulled down' towards the x-axis ($\sqrt{100} = 10$). If $y < 1$ then the points get pushed further away from the x-axis ($\sqrt{\frac{1}{4}} = \frac{1}{2}$).
  4. Vertical asymptotes on y remain vertical asymptotes on y^2.
  5. Horizontal asymptotes above the x-axis ($y = k$, say) become horizontal asymptotes $y = \pm\sqrt{k}$.
  6. Any points where the original curve hits the x-axis are also invariant ($\sqrt{0} = 0$). Also, the gradient of any points where y hits the x-axis become vertical on the y^2 graph.
  7. Any stationary point above the x-axis on y ($(2, 16)$, say) remain stationary points at $(2, \pm 4)$, say.

Polar Coordinates

- Polar coordinates are given as points with (r, θ) with the constraints $r \geqslant 0$ and (usually) either $0 \leqslant \theta < 2\pi$ or $-\pi < \theta \leqslant \pi$. The distance from the origin is r and θ is the angle made with the initial line (i.e. the positive x-axis) measured anti-clockwise. The angle constraints are used so that each point in space has a unique angle[6]. The 'pole' is sometimes used to describe the origin of your xy-grid in the context of a polar graph.

- Circles are described by 'r = constant'. Lines running out from the pole are described by 'θ = constant'.

- Polar curves are a new way of describing curves by showing a relationship between r and θ. They are usually given in the form $r = f(\theta)$.

- To sketch a polar curve $r = f(\theta)$ there are various tools to help you (in most cases completely analogous to the tools have help you draw $y = f(x)$).

  1. If in doubt, throw θ values into your $r = f(\theta)$ and work out r-values for given θ-values and plot them.
  2. If you are trying to draw $r = f(\theta)$, some students find it helpful to draw $y = f(x)$ to discover the general behavior of f.
  3. Understand that solving $\frac{dr}{d\theta} = 0$ gives you points where the curve is locally closest or furthest from the pole. If $\frac{d^2r}{d\theta^2} > 0$ then it is a point closest to the pole. If $\frac{d^2r}{d\theta^2} < 0$ then it is a point furthest from the pole.
  4. Solving $r = 0$ will give you the θ values (θ_1, say) that represent where the curve drops into the pole. Therefore the line $\theta = \theta_1$ will represent a tangent to the curve.

[6] Otherwise $(3, 0)$, $(3, 2\pi)$, $(3, 4\pi)$, ... would all be the same point.

5. Look for symmetries in your function. For example $r = \sin\theta$. We know the sine wave has symmetry about $\frac{\pi}{2}$; i.e. $\sin(\pi - \theta) \equiv \sin\theta$. Therefore the line $\theta = \frac{\pi}{2}$ must represent a line of symmetry on the polar curve.

   More generally if you can show that $f(2\alpha - \theta) \equiv f(\theta)$ then $\theta = \alpha$ represents a line of symmetry on the polar curve $r = f(\theta)$.

- To find areas on a polar graph we use the formula

$$\text{Area} = \int_{\theta_1}^{\theta_2} \tfrac{1}{2} r^2 \, d\theta.$$

- To convert from cartesian form to polar form use the relationships

$$x^2 + y^2 = r^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta, \qquad \tan\theta = \tfrac{y}{x}.$$

  These can be derived easily from the point (x, y) drawn with a right angled triangle to the origin.

Hyperbolic Functions

- Know that the hyperbolic trigonometric functions[7] are defined

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}, \qquad \sinh x \equiv \frac{e^x - e^{-x}}{2}, \qquad \tanh x \equiv \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

  Similarly (as you would expect) we define

$$\operatorname{sech} x \equiv \frac{1}{\cosh x}, \qquad \operatorname{cosech} x \equiv \frac{1}{\sinh x}, \qquad \coth x \equiv \frac{1}{\tanh x}.$$

- Two important relationships that drop out instantly are

$$\cosh x + \sinh x = e^x \qquad \text{and} \qquad \cosh x - \sinh x = e^{-x}.$$

- You must know (or, better yet be able to work out from the definitions) the sketches for all 6 hyperbolic curves. Also $\sinh x$ is 'one-to-one' so can be inverted without restricting the domain. However, $\cosh x$ is 'many-to-one' so a domain restriction is required ($x \geqslant 0$) to invert it.

- Differentiating the above definitions we quickly find

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sinh x = \cosh x.$$

- We also find $\cosh^2 x - \sinh^2 x = 1$ and $\sinh 2x = 2\sinh x\cosh x$. To derive results like these, run back to the exponential definitions and work from one side to the other. For example to prove the latter of the two results stated, start with $2\sinh x\cosh x$:

$$\begin{aligned} 2\sinh x\cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{(e^x - e^{-x})(e^x + e^{-x})}{2} \\ &= \frac{e^{2x} + 1 - 1 - e^{-2x}}{2} = \sinh 2x. \end{aligned}$$

[7]Compare with normal trig functions $\cos x \equiv \frac{e^{ix}+e^{-ix}}{2}$, $\sin x \equiv \frac{e^{ix}-e^{-ix}}{2i}$, $\tan x \equiv \frac{\sin x}{\cos x}$.

- You need to know the logarithmic forms[8] for the inverse hyperbolic functions:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \qquad \text{for all } x,$$
$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \qquad \text{for } x \geqslant 1,$$
$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \text{for } -1 < x < 1.$$

- For example: Solve $24\cosh x + 16\sinh x = 21$. Re-write the $\sinh x$ and $\cosh x$ in terms of e^x and then solve the resulting 'quadratic in disguise'.

$$\begin{aligned} 12(e^x + e^{-x}) + 8(e^x - e^{-x}) &= 21, \\ 20e^x - 21 + 4e^{-x} &= 0, \\ 20(e^x)^2 - 21(e^x) + 4 &= 0, \\ (5e^x - 4)(4e^x - 1) &= 0. \end{aligned}$$

This then solves to $x = \ln\frac{4}{5}$ or $x = \ln\frac{1}{4}$.

- A particularly useful identity which helps in some tougher problems is $(x - \sqrt{x^2-1})(x + \sqrt{x^2-1}) \equiv 1$. So

$$\frac{1}{x - \sqrt{x^2-1}} \equiv x + \sqrt{x^2-1} \qquad \text{and} \qquad \frac{1}{x + \sqrt{x^2-1}} \equiv x - \sqrt{x^2-1}.$$

Differentiation & Integration

- In C3 you will have seen the wonderful trick to find the derivative of $\ln x$:

$$\begin{aligned} y &= \ln x \\ e^y &= x \\ e^y &= \frac{dx}{dy} \\ \frac{1}{e^y} &= \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{x}. \end{aligned}$$

We can use the same trick for inverse trig functions:

$$\begin{aligned} y &= \sin^{-1} x \\ \sin y &= x \\ \cos y &= \frac{dx}{dy} \\ \frac{1}{\cos y} &= \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

[8] Derived by considering $y = \sinh^{-1} x$, $\Rightarrow$ $\sinh y = x$, $\Rightarrow$ $e^y - \frac{1}{e^y} = 2x$, $\Rightarrow$ $(e^y)^2 - 2x(e^y) - 1 = 0$. Then solve the resulting 'quadratic in disguise' for e^y.

Similarly we can derive the following important results (you should do so for yourself!):

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2},$$
$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}, \qquad \frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}, \qquad \frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}.$$

- A glance at the formula book[9] shows that the above six derivations yield results such as $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$. However, you should not allow yourself to get tied down to the formula book. I am a firm believer that the formula booklet should act as a guide only; showing you what substitution to use. For example if we needed to find $\int \frac{5}{9+4x^2}\,dx$ we would be lost with only the formula book, because it is not in precisely the same form. However it 'looks like' the '$\tan^{-1}$' answer in the formula book, so this is the hint to use a 'tan' substitution. Here we want $4x^2 = 9\tan^2\theta$; or, more simply, $2x = 3\tan\theta$. So (deep breath!)

$$\begin{aligned}
&\int \frac{5}{9+4x^2}\,dx && 2x = 3\tan\theta \\
=&\int \frac{5}{9+9\tan^2\theta}\frac{3}{2}\sec^2\theta\,d\theta && 2\,dx = 3\sec^2\theta\,d\theta \\
=&\frac{15}{2}\int \frac{\sec^2\theta}{9\sec^2\theta}\,d\theta \\
=&\frac{5}{6}\theta + c = \frac{5}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c.
\end{aligned}$$

- Completing the square is another useful thing to look for. Here I don't necessarily mean the strict C1 method where $9x^2+6x-15$ becomes $9(x+\frac{1}{3})^2-16$. A much more useful form for the former is $(3x+1)^2-16$, keeping everything in integers. So if asked to work out $\int \frac{7}{\sqrt{9x^2+6x-15}}\,dx$ we can re-write it as $\int \frac{7}{\sqrt{(3x+1)^2-16}}\,dx$. This looks very similar to the '$\cosh^{-1}$' differential above. Therefore we need a 'cosh' substitution. Here we want $(3x+1)^2 = 16\cosh^2 u$; or, more simply, $3x+1 = 4\cosh u$. So (here we go!)

$$\begin{aligned}
&\int \frac{7}{\sqrt{(3x+1)^2-16}}\,dx && 3x+1 = 4\cosh u \\
=&\int \frac{7}{\sqrt{16\cosh^2 u - 16}}\frac{4}{3}\sinh u\,du && 3\,dx = 4\sinh u\,du \\
=&\frac{28}{3}\int \frac{\sinh u}{4\sinh u}\,du \\
=&\frac{7}{3}u + c = \frac{7}{3}\cosh^{-1}\left(\frac{3x+1}{4}\right) + c.
\end{aligned}$$

- Another useful trick is to split the numerator of a fraction in an integral into two bits of more use. For example, if faced with $\int \frac{2x+3}{x^2+4x+1}\,dx$ you can split it into $\int \frac{2x+4}{x^2+4x+1} - \frac{1}{x^2+4x+1}\,dx$, each bit of which is now more easily handled.

- A useful substitution for integrals that involve trigonometric functions is $t = \tan\left(\frac{x}{2}\right)$. This is a boon because it changes horrible integrals with trig functions into new integrals with no trig

[9] If your school has not provided you with a copy, you should ask for (demand) one. It is very useful to know what's in it. However, if you're going to an interview at a top university and you say to your interviewer "I would have to look at a formula book to answer that" then you can expect a rejection letter soon after.

at all[10]. Given this substitution it can be shown that

$$\tan x = \frac{2t}{1-t^2}, \qquad \sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}.$$

These must be learnt! When applying this, you must also use the fact that $\frac{dx}{dt} = \frac{2}{1+t^2}$, to replace the '$dx$' at the end of the integral by '$\frac{2}{1+t^2}\,dt$'. For example to evaluate $\int \frac{\sin x}{1+\cos x}\,dx$ we find

$$\begin{aligned}\int \frac{\sin x}{1+\cos x}\,dx &= \int \frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt \qquad \text{using } t = \tan\left(\frac{x}{2}\right)\\ &= \int \frac{2t}{1+t^2}\,dt\\ &= \ln(1+t^2)+c\\ &= \ln(1+\tan^2\left(\frac{x}{2}\right))+c.\end{aligned}$$

The only thing to add is that if you were faced with $\int \frac{\sin 10x}{1+\cos 10x}\,dx$ the substitution would be $t = \tan 5x$. This would change the 'dx' replacement (by the chain rule) to '$\frac{1}{5(1+t^2)}\,dt$'. Therefore

$$\int \frac{\sin 10x}{1+\cos 10x}\,dx = \int \frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\frac{1}{5(1+t^2)}\,dt = \frac{1}{10}\int \frac{2t}{1+t^2}\,dt.$$

Make sure you 'get' this; it is a little subtle.

Reduction Formulae

- Reduction formulae involve integrals which do not only involve x, but also n. In general we write

  $$I_n = \int (\text{something to do with } x \text{ and } n)\,dx$$

  to indicate that the integral depends on n. The aim (usually) is to find a relationship between I_n and I_{n-1} or a relationship between I_n and I_{n-2} and then use this relationship to evaluate a specific integral (I_6, say). Integration by parts tends to be the method needed to find such a relationship since the integration by parts formula[11] contains an integral on each side of the equation which may be manipulated into the desired relationship. You do occasionally need to be quite cunning![12]

- For example find $\int_0^1 x^6 e^{2x}\,dx$. Clearly[13] the hint is to let $I_n = \int_0^1 x^n e^{2x}\,dx$. By parts

  $$\begin{aligned}I_n &= \int_0^1 x^n e^{2x}\,dx = \left[\frac{x^n e^{2x}}{2}\right]_0^1 - \frac{n}{2}\int_0^1 x^{n-1}e^{2x}\,dx,\\ I_n &= \frac{e^2}{2} - \frac{n}{2}I_{n-1}.\end{aligned}$$

[10] I don't know about you, but I *hate* trig integrals and the sooner I can get rid of the trig bits the better.

[11] $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$.

[12] Remember the Alloway special!

[13] Hopefully you can see why letting $I_n = \int_0^1 x^7 e^{nx}\,dx$ is a dreadful idea!

Now we have the relationship between I_n and I_{n-1} we need some 'low' integral that we can evaluate easily: I_0 fits the bill since $I_0 = \int_0^1 e^{2x}\,dx = \frac{e^2-1}{2}$. So

$$\begin{aligned}
I_6 &= \frac{e^2}{2} - \frac{6}{2}I_5\\
&= \frac{e^2}{2} - \frac{6}{2}\left(\frac{e^2}{2} - \frac{5}{2}I_4\right)\\
&= \text{...work through for yourself...}\\
&= \frac{7e^2}{8} - \frac{45}{8}.
\end{aligned}$$

- 'Snapping off' bits of trig functions often helps (i.e. writing $\sin^n x$ as either $\sin x \sin^{n-1} x$ or $\sin^2 x \sin^{n-2} x$). For example find a reduction formula for $I_n = \int \cos^n x\,dx$. Snap off a '$\cos x$' and then do parts, integrating the $\cos x$ and differentiating the $\cos^{n-1} x$.

$$\begin{aligned}
I_n = \int \cos^n x\,dx &= \int \cos x \cos^{n-1} x\,dx\\
&= \sin x \cos^{n-1} x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx\\
&= \sin x \cos^{n-1} x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx\\
I_n &= \sin x \cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n.
\end{aligned}$$

Isolating I_n we find $I_n = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}I_{n-2}$. We can find I_0 and I_1 easily enough, which means we can evaluate $I_n = \int \cos^n x\,dx$ for any positive integer n.

Maclaurin Series

- The Maclaurin series/expansion for a function is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!}x^r.$$

This is a remarkable formula; it implies that you can know a function completely over *all* values of x provided you know all the derivatives of a function at *one* value of x.

- You must know (and any good candidate ought to derive for themselves) the following standard expansions:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots &&\text{valid for all } x,\\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots &&\text{valid for all } x,\\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots &&\text{valid for all } x,\\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots &&\text{valid for } -1 < x < 1.
\end{aligned}$$

It is good to note that the e^x series differentiates to itself and the $\sin x$ and $\cos x$ series differentiate twice to minus themselves (as they should). Also we note that if we differentiate $\ln(1+x)$ we get $\frac{1}{1+x}$ and the general binomial expansion of this (using C4 methods) is precisely what we get by differentiating our Maclaurin expansion[14].

[14] $1 - x + x^2 - x^3 + \ldots$

- You rarely (if ever) need to derive a Maclaurin series from first principles[15]. What you need to do is apply the series in the 'formula booklet' to similar situations.

- For example find the Maclaurin series for $\frac{3\cos(2x)}{1+\ln(1-4x)}$. So

$$\begin{aligned}\frac{3\cos(2x)}{1+\ln(1-4x)} &= \frac{3\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\dots\right)}{1+\left((-4x)-\frac{(-4x)^2}{2}+\frac{(-4x)^3}{3}-\dots\right)}\\ &= \frac{3-6x^2+2x^4-\dots}{1-4x-8x^2+\dots}\\ &= 3+12x+66x^2+\dots\end{aligned}$$

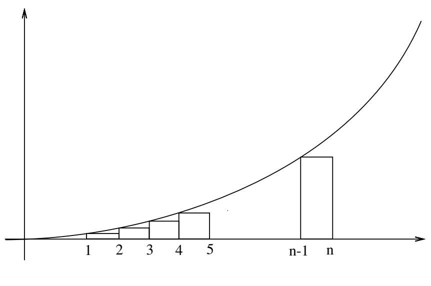
To do the last step you consider the general binomial expansion on $(1-4x-8x^2+\dots)^{-1}$.

Series & Integrals

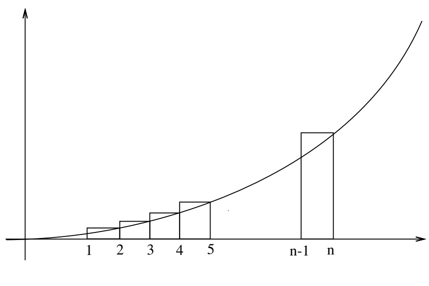
- You must be able to sandwich certain integrals between sums ($\text{sum}_1 < \text{integral} < \text{sum}_2$) and sandwich certain sums between integrals ($\text{integral}_1 < \text{sum} < \text{integral}_2$). The first of these is easier to formulate, but the second of these is more useful since you are currently better at integrals than sums.

- You must be abundantly clear whether you are dealing with an *increasing* or *decreasing* function and you must always draw a sketch of the relevant curve and associated rectangles to make sure you are not writing gibberish (as I have occasionally done in class). Remember a function is increasing if $\frac{dy}{dx} \geqslant 0$ for all x-values in a range. Similarly a function is decreasing if $\frac{dy}{dx} \leqslant 0$.

- For example sandwich $\int_1^n x^3\,dx$ between two sums. Firstly $y = x^3$ is an increasing function in the range stated, so if we want the sum below the integral we want the rectangles where the left height joins the curve. So $1^3 + 2^3 + \cdots + (n-1)^3 < \int_1^n x^3\,dx$. Similarly the sum above the integral is where the right height joins the curve.



and



So

$$\begin{aligned}1^3+2^3+\cdots+(n-1)^3 \;&<\; \int_1^n x^3\,dx \;<\; 2^3+3^3+\cdots+n^3,\\ \sum_{i=1}^{n-1} i^3 \;&<\; \int_1^n x^3\,dx \;<\; \sum_{i=2}^{n} i^3.\end{aligned}$$

- For example sandwich $\int_1^n \frac{1}{x^2}\,dx$ between two sums. Here we have a decreasing function in the range required, so the lower limit is now given by the rectangles whose right heights join

[15] That means you McKelvie!

the curve. You should therefore find

$$\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \cdots + \left(\tfrac{1}{n}\right)^2 \;<\; \int_1^n \frac{1}{x^2}\,dx \;<\; \left(\tfrac{1}{1}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \cdots + \left(\tfrac{1}{n-1}\right)^2,$$

$$\sum_{i=2}^{n}\left(\tfrac{1}{i}\right)^2 \;<\; \int_1^n \frac{1}{x^2}\,dx \;<\; \sum_{i=1}^{n-1}\left(\tfrac{1}{i}\right)^2.$$

- Notice the way the limits on the sums seem to *flip* between increasing and decreasing functions.

- Sandwiching a sum between two integrals is a little more fiddly. For example sandwich $\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \cdots + \sqrt[3]{n}$ between two integrals. Clearly the function we are considering here is $y = \sqrt[3]{x}$; this is an increasing function in the range. By considering two suitable sketches we find

$$\int_0^n \sqrt[3]{x}\,dx \;<\; \sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \cdots + \sqrt[3]{n} \;<\; \int_1^{n+1} \sqrt[3]{x}\,dx,$$

$$\frac{3n^{4/3}}{4} \;<\; \sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \cdots + \sqrt[3]{n} \;<\; \frac{3(n+1)^{4/3} - 3}{4}.$$

- Similarly if you want to sandwich $f(1) + f(2) + \cdots + f(n)$ between two integrals where $y = f(x)$ is a *decreasing* function for $1 < x < n$ you should find

$$\int_1^{n+1} f(x)\,dx \;<\; f(1) + f(2) + \cdots + f(n) \;<\; \int_0^n f(x)\,dx.$$

  Draw a sketch to see why.

Numerical Methods

- In C3 you considered iterations of the form $x_{n+1} = F(x_n)$ which give a progression of values $x_0, x_1, x_2, \ldots x_i, \ldots x_n \ldots$ which hopefully *converge* towards a solution of the equation $x = F(x)$. The 'true value' of the solution is denoted α. We define the *error* at any point of the iteration to be the difference between the 'true value' and the value of the iteration at that point; i.e.

$$e_n = \alpha - x_n \qquad \text{or} \qquad e_i = \alpha - x_i$$

  It is obviously to be hoped that e_i's get smaller as the iteration progresses.

- [Taylor series (which are not technically on the FP2 syllabus, but are needed for a full understanding of what follows) are a generalisation of Maclaurin series. Whereas Maclaurin series are 'centred around' $x = 0$ and provide increasingly good approximations to a function around the y-axis, Taylor series provide approximations to a function around any x-value you choose. The Taylor expansion around $x = a$ is

$$F(x) = F(a) + F'(a)(x-a) + \frac{F''(a)}{2!}(x-a)^2 + \frac{F'''(a)}{3!}(x-a)^3 + \cdots.$$

- When considering the iteration $x_{n+1} = F(x_n)$ we can Taylor expand $F(x_n)$ about the root α, so

$$x_{n+1} = F(x_n) = F(\alpha) + F'(\alpha)(x_n-\alpha) + \frac{F''(\alpha)}{2!}(x_n-\alpha)^2 + \frac{F'''(\alpha)}{3!}(x_n-\alpha)^3 + \cdots.$$

  But note that $F(\alpha) = \alpha$ because we are solving the equation $x = F(x)$. Therefore $x_{n+1} = \alpha + F'(\alpha)(x_n-\alpha) + \frac{F''(\alpha)}{2!}(x_n-\alpha)^2 + \frac{F'''(\alpha)}{3!}(x_n-\alpha)^3 + \cdots.$]

- If $\boxed{F'(\alpha) \neq 0}$ and we are in the neighbourhood of (i.e. close to) the root we can truncate the Taylor series at the $(x_n - \alpha)$ term to obtain

$$x_{n+1} \approx \alpha + F'(\alpha)(x_n - \alpha).$$

Rearranging we find $\alpha - x_{n+1} \approx F'(\alpha)(\alpha - x_n)$ which gives $e_{n+1} \approx F'(\alpha)e_n$ so

$$\boxed{\frac{e_{n+1}}{e_n} \approx F'(\alpha) \approx \text{constant.}}$$

This shows that we require $-1 < F'(\alpha) < 1$ to get the desired *convergence* because we need the errors to get smaller as we iterate.

- If $\boxed{F'(\alpha) = 0}$ the second term in the Taylor series vanishes which means we need the next term, so

$$x_{n+1} \approx \alpha + \frac{F''(\alpha)}{2!}(x_n - \alpha)^2.$$

This rearranges to $\alpha - x_{n+1} \approx -\frac{F''(\alpha)}{2!}(\alpha - x_n)^2$ and so $e_{n+1} \approx -\frac{F''(\alpha)}{2!}e_n^2$ and therefore

$$\boxed{e_{n+1} \propto (e_n)^2;}$$

this is called *quadratic convergence* and these iterations converges quickly because if e_n is small (close to the root) then $(e_n)^2$ is much smaller (e.g. $0.01^2 = 0.0001$).

- The Newton-Raphson Method for numerical solution of equations is an ingenious method which takes the tangent to a curve at a point and uses its x-axis intercept as the next value for the iteration. For a given start value $x = x_1$ the iteration is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This is derived thus:

  - Start with x_n,
  - Go to the curve at this point $(x_n, f(x_n))$,
  - Construct tangent using $f'(x_n)$ as the gradient and $y - y_1 = m(x - x_1)$,
  - $y - f(x_n) = f'(x_n)(x - x_n)$,
  - Put $y = 0$ to find where tangent crosses x-axis,
  - This x value is x_{n+1}.

Newton-Raphson converges quadratically[16] explaining its speed.

[16]The N-R iteration $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ can be thought of as $x_{n+1} = F(x_n)$ and we wish to show that $F'(\alpha) = 0$ for quadratic convergence. So differentiating $F(x)$ we find

$$\begin{aligned} F'(x) &= \frac{d}{dx}(F(x)) \\ &= \frac{d}{dx}\left(x - \frac{f(x)}{f'(x)}\right) \\ &= 1 - \left(\frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}\right) \text{ (by the quotient rule)} \\ &= \frac{f(x)f''(x)}{[f'(x)]^2} \end{aligned}$$

Putting in $x = \alpha$ we obtain $F'(\alpha) = \frac{f(\alpha)f''(\alpha)}{[f'(\alpha)]^2}$. However $f(\alpha)$ must be zero because we are solving $f(x) = 0$ and α is a root. Therefore $F'(\alpha) = 0$ when using N-R $\Rightarrow$ quadratic convergence.

Differential Equations

First Order

- Given a first order differential equation, first see if it is separable *à la* C4.

$$\frac{dy}{dx} = f(x)g(y) \qquad \Rightarrow \qquad \int \frac{dy}{g(y)} = \int f(x)\,dx.$$

  Remember to add the "+c" to one side only. If you leave the "+c" unevaluated then your answer represents the general solution; otherwise your answer is a particular solution. (This will hardly ever happen in FP3.) In these notes $P \equiv P(x)$ and $Q \equiv Q(x)$.

- Differential equations can sometimes be changed by a substitution into a more accessible form for solution. Usually this will be trivial, but sometimes replacing the differentiable bit will require the chain/product rules, and can be a little fiddly. For example find the general solution of

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \text{ by the substitution } y = xu.$$

  Clearly the question is guiding us to get rid of all the y's in the original equation so we find

$$\begin{aligned}\frac{d(xu)}{dx} &= \frac{x^2+(xu)^2}{2x(xu)},\\ u + x\frac{du}{dx} &= \frac{1+u^2}{2u},\\ x\frac{du}{dx} &= \frac{1-u^2}{2u}.\end{aligned}$$

  This is separable, and we obtain the solution $\frac{1}{1-u^2} = Ax$ for some arbitrary A. Eliminating u from this using $y = xu$ we find $y^2 = x^2 - kx$ for some constant k.

- If it is not separable and there is no substitution to try then try to re-arrange into the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

  You will then need to multiply by an integrating factor (IF) which is defined as IF $= e^{\int P\,dx}$. Then we can[17] re-write the LHS in the form $d(\ldots)/dx$.

- Example solve the differential equation $x\frac{dy}{dx} + 2y = \frac{4}{x}$. We rearrange to get $\frac{dy}{dx} + \frac{2}{x}y = \frac{4}{x^2}$ the we work out the IF to be $e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$ so the equation becomes $x^2\frac{dy}{dx} + 2xy = 4$. We can now write the LHS to obtain

$$\frac{d(x^2y)}{dx} = 4 \quad \Rightarrow \quad \int d(x^2y) = \int 4\,dx \quad \Rightarrow \quad x^2y = 4x + c \quad \Rightarrow \quad y = \frac{4}{x} + \frac{c}{x^2}.$$

  This represents the general solution of the differential equation. If we were told that $y = 6$ when $x = 1$ we would put this into the GS and find $c = 2$. This would give us a particular solution of $y = \frac{4}{x} + \frac{2}{x^2}$.

- *Homogeneous linear* differential equations are of the form $\frac{dy}{dx} + ay = 0$ or $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ for constant a and b.

  Non-homogeneous linear differential equations are of the form $\frac{dy}{dx} + ay = f(x)$ or $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$ for constant a and b.

[17] By understanding that $\frac{d}{dx}(ye^{\int P\,dx}) = e^{\int P\,dx}\frac{dy}{dx} + Pe^{\int P\,dx}y$.

- Linear (and *only* linear) differential equations can be approached by the *auxiliary equation method*: we try a solution of the form $y = Ae^{\lambda x}$ in the original equation made homogeneous and construct an auxiliary equation in λ. We can then find λ and this will give us a complementary function (CF). For example $\frac{dy}{dx} - 3y = 2x$ would be modified to $\frac{dy}{dx} - 3y = 0$. Then the AE would be $\lambda Ae^{\lambda x} - 3Ae^{\lambda x} = 0$ so $\lambda = 3$ so the CF would be $y = Ae^{3x}$. If the original equation was homogeneous then the CF *is* the GS of the equation, but if it is non-homogeneous then you need to find particular integral (PI) which we add to the CF to get the GS. GS = CF + PI.

- In the above example we need to find a particular integral (PI) based on the form of $f(x)$ which in this case is $2x$. You guess the type of it depending on the type of $f(x)$. Here is a table of logical trials:

$f(x)$	TRIAL
linear	$lx + m$
polynomial order n	$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
trig functions involving sin or cos px	$l \sin px + m \cos px$
exponential involving e^{px}	ce^{px}

So here we try a PI of $y = mx + n$. Putting it in we find $m - 3(mx + n) = 2x$ and equating coefficients we find $m = -\frac{2}{3}$ and $n = -\frac{2}{9}$. The general solution (GS) would then be

$$\text{GS} = \text{CF} + \text{PI} \qquad \Rightarrow \qquad y = Ae^{3x} - \tfrac{2}{3}x - \tfrac{2}{9}.$$

Second Order

- Given a second order DE of the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

for constant a and b you construct the auxiliary equation[18] (AE) $\lambda^2 + a\lambda + b = 0$. The solution to the AE dictates the form of the complementary function (CF). There are three cases to consider depending on the type of solutions the AE has:

	TYPE AE		CF
1.	Real and distinct roots, λ_1 and λ_2	$\Rightarrow$	$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
2.	Repeated real root, α.	$\Rightarrow$	$y = e^{\alpha x}(A + Bx)$
3.	Complex roots, $\alpha \pm i\beta$	$\Rightarrow$	$y = e^{\alpha x}(A \sin \beta x + B \cos \beta x)$

Notice that each CF has two arbitrary constants A and B. This makes sense, because we are effectively integrating twice and so would expect this[19].

- If you are dealing with a homogeneous equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ then, as with first order, the GS is just the CF and you are done (subject to boundary/initial conditions!).

- If you are dealing with a non-homogeneous equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$ then we need to find a particular integral (PI) with the same principles as above. For example solve the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin x + \cos x.$$

[18] This is not plucked out of thin air! The AE is obtained by trying a solution of the form $y = Ae^{\lambda x}$ for constants A and λ.

[19] The number of arbitrary constants should always match the order of the equation.

This gives AE of $\lambda^2 - 3\lambda + 2 = 0$ so $\lambda = 1$ or $\lambda = 2$. Therefore the CF is $y = Ae^x + Be^{2x}$. Looking at $\sin x + \cos x$ we would clearly try $y = m \sin x + n \cos x$ as our PI. Putting this in we discover

$$\begin{aligned}(-m \sin x - n \cos x) - 3(m \cos x - n \sin x) + 2(m \sin x + n \cos x) &= \sin x + \cos x,\\ (-m + 3n + 2m) \sin x + (-n - 3m + 2n) \cos x &= \sin x + \cos x.\end{aligned}$$

Equating coefficients of $\sin x$ and $\cos x$ we discover $m + 3n = 1$ and $n - 3m = 1$. This solves to $m = -\frac{1}{5}$ and $n = \frac{2}{5}$ giving us a GS of

$$y = Ae^x + Be^{2x} - \tfrac{1}{5} \sin x + \tfrac{2}{5} \sin x.$$

- The only *caveat* to the PI guesses is if the trial function for a PI is the same as one of CFs: you then multiply the trial function by x. For example find the general solution of

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}.$$

The auxiliary equation gives $\lambda = 2$ or $\lambda = 3$. This gives a CF of $y = Ae^{2x} + Be^{3x}$. We would normally try a PI of $y = me^{2x}$, but we notice that e^{2x} is part of the CF, so we try a PI of $y = mxe^{2x}$ instead. Put this into the original differential equation and we find

$$\begin{aligned}(2me^{2x} + 2me^{2x} + 4mxe^{2x}) - 5(me^{2x} + 2mxe^{2x}) + 6mxe^{2x} &= e^{2x},\\ 2m + 2m + 4mx - 5m - 10mx + 6mx &= 1,\\ m &= -1.\end{aligned}$$

Therefore the GS is $y = Ae^{2x} + Be^{3x} - xe^{2x}$.

- If you are given conditions to be satisfied by the system at the start, then you find the GS and then put in the information to find the value of the arbitrary constants.

Vectors

- Recall that a line through position vector $\mathbf{a}$ and with direction $\mathbf{d}$ is, in vector form, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$. Recall also that the line through position vectors $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$ or $\mathbf{r} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b})$ or equivalent. This is because when I subtract two position vectors $(\mathbf{b} - \mathbf{a})$ it yields the translation vector that travels from $\mathbf{a}$ to $\mathbf{b}$. For example find the line that passes through $(1, 3, 4)$ and $(3, 1, 8)$: This gives $\mathbf{r} = \begin{pmatrix}1\\3\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\-2\\4\end{pmatrix}$ which would then simplify to give $\mathbf{r} = \begin{pmatrix}1\\3\\4\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$. Always look to simplify the direction vector if possible.

- A line can also be given in cartesian form

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}.$$

To convert cartesian to vector form place each of the three elements equal to λ and 'unwrap'. For example $\frac{x-3}{2} = \frac{y+7}{2} = \frac{2-z}{3}$: so

$$\frac{x-3}{2} = \lambda \qquad \frac{y+7}{2} = \lambda \qquad \frac{2-z}{3} = \lambda.$$

So $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3+2\lambda\\-7+2\lambda\\2-3\lambda\end{pmatrix}$, therefore $\mathbf{r} = \begin{pmatrix}3\\-7\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\-3\end{pmatrix}$. To convert the other way should be trivial; unwind the above (practice for yourself).

- A plane in 3D can be given by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{p} + \mu\mathbf{q}$. You can think of this as a point $\mathbf{a}$ being stretched along the line with direction $\mathbf{p}$ and then that line being stretched along in direction $\mathbf{q}$ to form a plane. This is the least helpful form for the plane (IMHO). [If I saw this I would cross $\mathbf{p}$ and $\mathbf{q}$ to get $\mathbf{n}$ and then use the form $ax + by + cz = k$: see below.]

- A plane in 3D through point $\mathbf{a}$ and normal $\mathbf{n}$ can be given by $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$ where $\mathbf{r}$ represents all the points on the plane[20]. This is because if the dot product is zero then $(\mathbf{r} - \mathbf{a})$ and $\mathbf{n}$ must be at right angles and $(\mathbf{r} - \mathbf{a})$ is the vector that travels from $\mathbf{a}$ to any point $\mathbf{r}$. A nice sketch in the textbook P250. This can then be written $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$ so $\begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = n_1x + n_2y + n_3z = \text{constant}$. Therefore...

- ...a plane can most usefully be in the form $ax + by + cz = k$ where $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector and k is some constant for *that* plane. As k varies it will produce parallel planes, like pages in a book.

- To find the intersection of a line and a plane is easy. Best done by example, find intersection of $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $2x + y - 3z = -24$. From the line we know that $x = 2 - t$ etc. so we place all of these into the plane and solve for t, so $2(2 - t) + (1 + t) - 3(3 + 3t) = -24$ which solves to give $t = 2$. Place this back into the line and the point is $(0, 3, 9)$.

- You also need to be able to determine whether a line lies *in* a plane or parallel to it. If you try to find where the line crosses the plane (like above) you will either boil your equation for λ down to a consistency $(1 = 1)$ in which case the line lies in the plane, or an inconsistency $(0 = 1)$ in which case the line is parallel to the plane.

- To find the shortest distance between a line and a point, do a sketch of the line and the point, and construct the triangle between the point away from the line (P), the point $\mathbf{a}$ on the line (A) and the point which is closest to the point (F). We want the length PF in the right angled triangle APF. The angle $P\widehat{A}F$ can be found by dotting the direction vector of the line with the vector $\overrightarrow{AP}$ and we can work out the length AP by working out the magnitude of $\overrightarrow{AP}$. Then it's just sin in a right angled triangle to get length PF.

- To find shortest distance between a point and a plane is a trivial extension of the above. Construct the line through the point with direction vector normal to the plane. Find where this line crosses the plane and then find the distance between these two points. [Quick reminder: The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

or, equivalently, the distance between points with position vectors $\mathbf{a}$ and $\mathbf{b}$ is the magnitude of $(\mathbf{b} - \mathbf{a})$.]

- Must be able to find the intersection of two planes. For example find intersection of $x+2y-2z = 2$ and $2x+3y-7z = 1$: Up to you which variable you would like to eliminate, but I'd do twice

[20] $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

the first minus the second. This gives $y + 3z = 3$. Let $z = t$ (you'd be silly to let $y = t$, try it to see why!) and we find $y = 3 - 3t$ and then $x = 2 - 2y + 2z = 2 - 2(3 - 3t) + 2(t) = -4 + 8t$. Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 + 8t \\ 3 - 3t \\ t \end{pmatrix} \qquad \text{so} \qquad \mathbf{r} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix}.$$

You could also have solved this by crossing the normals of the planes to discover the direction vector of the line of intersection and then find any point where the planes cross.

- The angle between two planes is the same as the angle between their normals. The angle between a plane and a line is $\frac{\pi}{2}$ minus the angle between the line and the plane's normal.

- The vector product of two vectors is defined

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \widehat{\mathbf{n}}, \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} bn - cm \\ cl - an \\ am - bl \end{pmatrix}.$$

  In practice you very rarely need the $\sin\theta$ bit of this definition (any angles you need are always more easily accessible by the dot product). It is most useful in the way it constructs a vector perpendicular to both original vectors.

- For example: Find the equation of the plane through $A(1, 2, -1)$,$B(2, 3, 4)$ and $C(-2, 0, 1)$. We construct the normal vector to the plane by crossing $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \left(\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right) \times \left(\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -17 \\ 1 \end{pmatrix}.$$

  Therefore the plane is $12x - 17y + z = \text{const}$. We find the constant by taking your favourite of A, B or C and plugging it in[21]. So $12x - 17y + z = -23$.

- The shortest distance between the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ is given by

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1).(\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}.$$

Complex Numbers

- Recall that the complex number $z = a + ib$ has modulus $|z| \equiv r = \sqrt{a^2 + b^2}$ and argument $\arg(z) = \theta = \tan^{-1}\frac{b}{a}$, where argument is measured anti-clockwise from the positive real axis. Also recall that arguments can be of either convention $0 \leqslant \arg(z) < 2\pi$ or $-\pi < \arg(z) \leqslant \pi$. This is purely arbitrary and will be clear from the question what they want. [In my mind arguments are such that $-\infty < \arg(z) < \infty$ and they can keep twirling round, but OCR forces each complex number to have a unique argument.]

- By considering a right angled triangle like the one at the top of P295 we discover $z = a + ib = r(\cos\theta + i\sin\theta)$. This is the *polar form* of a complex number.

  It can also be shown[22] that $r(\cos\theta + i\sin\theta) \equiv re^{i\theta}$. This is the *exponential form* of a complex number and is unbelievably useful![23]

[21] A nice check is to put more than one point in and check that you get the same constant each time.

[22] By mucking about with $\cos\theta + i\sin\theta = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots) + i(x - \frac{x^3}{3!} + \ldots) = \text{lots of working} = e^{i\theta}$.

[23] It should be noted that an alternative notation for e^x at a higher level is $\exp(x)$. This is because the power on the e can sometimes become quite complicated.

- Properties of complex number multiplication and division become immediately apparent:

$$re^{i\alpha} \times \rho e^{i\beta} = (r\rho)e^{i(\alpha+\beta)} \quad \text{When multiplying, add arguments and multiply moduli,}$$
$$\frac{re^{i\alpha}}{\rho e^{i\beta}} = \left(\frac{r}{\rho}\right)e^{i(\alpha-\beta)} \quad \text{When dividing, subtract arguments and divide moduli.}$$

So given a complex number w, if I were to multiply w by the complex number $2e^{\frac{i\pi}{3}}$ (say) then the result would be the complex number with twice the length/modulus of w and rotated $\frac{\pi}{3}$ anti-clockwise from the original w. This is called a spiral-enlargement.

- De Moivre's Theorem[24] states that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad \text{or} \qquad (e^{i\theta})^n = e^{in\theta}.$$

For integer n this can be proven by induction. The reason for the truth of De Moivre's theorem should be obvious from the above two properties for multiplication and division. To raise complex number w to the power four (say) the modulus would be raised to the power four, but the argument would be made four times bigger…which is what De Moivre says.

We notice the special case

$$(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta \qquad \text{or} \qquad (e^{i\theta})^{-1} = e^{-i\theta}$$

which comes up a lot; the inverse of a complex number with unit modulus is its complex conjugate.

- You can use De Moivre's theorem to derive certain trigonometric results. You need

$$\cos n\theta = \text{Re}((\cos\theta + i\sin\theta)^n) \qquad \text{and} \qquad \sin n\theta = \text{Im}((\cos\theta + i\sin\theta)^n).$$

You can then use the standard relationship $\sin^2\theta + \cos^2\theta = 1$ to manipulate any raw results. Your algebra needs to be top-notch here; any slip at the start of a question can cost you dearly on a multi-parter. For example express $\cos 5\theta$ in terms of powers of $\cos\theta$. So

$$\begin{aligned}\cos 5\theta &= \text{Re}[(\cos\theta + i\sin\theta)^5]\\ &= \text{Re}[\cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5]\\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\ &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.\end{aligned}$$

- For problems involving $\tan n\theta$ you would consider the expansions of $\sin n\theta$ and $\cos n\theta$ and use $\tan n\theta \equiv \dfrac{\sin n\theta}{\cos n\theta}$. For example

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{\text{Im}[(\cos\theta + i\sin\theta)^3]}{\text{Re}[(\cos\theta + i\sin\theta)^3]} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}.$$

You could then divide both numerator and denominator by $\cos^3\theta$ to obtain the 'nicer'

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

[24] Abraham de Moivre, an 18th century statistician and consultant to gamblers. French…

- Given any complex number of unit modulus $z = e^{i\theta} = \cos\theta + i\sin\theta$ it can easily be shown that
$$\frac{1}{2}\left(z + \frac{1}{z}\right) = \cos\theta \qquad \text{and} \qquad \frac{1}{2i}\left(z - \frac{1}{z}\right) = \sin\theta.$$
Similarly we can derive (by using De Moivre on the unit complex number; $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$) the very useful relations
$$\frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = \cos n\theta \qquad \text{and} \qquad \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right) = \sin n\theta.$$
So whereas we can use De Moivre to find multiple angle expressions (such as $\sin 6\theta$) in terms of powers of sin and cos, we can use the above to write powers of sin and cos (such as $\sin^7\theta$) in terms of multiple angles. For example express $\cos^6\theta$ as a sum of multiple angles of $\cos\theta$.
$$\begin{aligned}\cos^6\theta &= \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^6,\\ &= \frac{1}{64}\left(z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}\right),\\ &= \frac{1}{64}\left(\left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20\right),\\ &= \frac{1}{64}\left(2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20\right),\\ &= \frac{1}{32}\left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right).\end{aligned}$$

- You could then use the above to help you with integrals:
$$\begin{aligned}\int \cos^6\theta\, d\theta &= \frac{1}{32}\int \left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right)\, d\theta,\\ &= \tfrac{1}{32}\left(\tfrac{1}{6}\sin 6\theta + \tfrac{3}{2}\sin 4\theta + \tfrac{15}{2}\sin 2\theta + 10\theta + c\right),\\ &= \frac{\sin 6\theta}{192} + \frac{3\sin 4\theta}{64} + \frac{15\sin 2\theta}{64} + \frac{5}{16} + c'.\end{aligned}$$

- You need to be able (using the ideas of roots of unity) to solve any equation of the form $z^n = a + ib$ (i.e. to get all n solutions to this equation). To get the *primary solution* you convert the $a + ib$ into the form $Re^{i\theta}$ and then obtain the first solution by raising both sides to the power $\frac{1}{n}$. So
$$\begin{aligned}z^n &= a + ib\\ z^n &= Re^{i\theta}\\ z &= (Re^{i\theta})^{1/n}\\ z &= \sqrt[n]{R}\, e^{\frac{i\theta}{n}}.\end{aligned}$$
This is the primary solution and then you find the others by using the fact they are all evenly spaced around the circle of radius $\sqrt[n]{R}$, like spokes on a bike. So you keep adding $\frac{2\pi}{n}$ on to the argument of $\sqrt[n]{R}e^{\frac{i\theta}{n}}$, $n - 1$ times to get all n solutions to the equation.

- For example solve $z^4 = -16$. We rewrite in the form $z^4 = 16e^{i\pi}$. Therefore the primary solution is $z = 2e^{\frac{i\pi}{4}}$. Adding on $\frac{2\pi}{4} = \frac{\pi}{2}$ to the arguments we find the four solutions
$$z = 2e^{\frac{i\pi}{4}}, 2e^{\frac{i3\pi}{4}}, 2e^{\frac{i5\pi}{4}}, 2e^{\frac{i7\pi}{4}}.$$

If need be you could then convert these back to $a + ib$ form and get:

$$z = (\sqrt{2} + i\sqrt{2}), (-\sqrt{2} + i\sqrt{2}), (-\sqrt{2} - i\sqrt{2}), (\sqrt{2} - i\sqrt{2}).$$

By taking the roots in complex conjugate pairs you could then factorise z^4+16 into the product of two real quadratic factors as

$$\begin{aligned} z^4 + 16 &= (z - (\sqrt{2} + i\sqrt{2}))(z - (\sqrt{2} - i\sqrt{2}))(z - (-\sqrt{2} + i\sqrt{2}))(z - (-\sqrt{2} - i\sqrt{2})) \\ &= (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4). \end{aligned}$$

Groups

- A group $(G, \circ)$ is a non-empty set G with a binary operation $\circ$ which
  - is **closed**, (for every a and b in G, $a \circ b$ also lies in G),
  - is **associative**, (for every a, b and c in G, $(a \circ b) \circ c = a \circ (b \circ c)$),
  - has a unique **identity** element, (an element e such that $e \circ a = a \circ e = a$ for all a in G),
  - every element has its own **inverse**, (for every a in G there exists a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = e$).

  A group can be represented as a Latin square. For example

	e	a	b
e	$e \circ e$	$e \circ a$	$e \circ b$
a	$a \circ e$	$a \circ a$	$a \circ b$
b	$b \circ e$	$b \circ a$	$b \circ b$

  Each row and column must contain every element of G once only. You can find the identity easily from this by looking for the row or column which is unchanged. Inverses are easy to find from a Latin Square; you merely look for which other element makes it the identity.

- If you are asked to show that something is a group in an exam you must tick off each of the above criteria one-by-one. For example show that the set $\{1, -1, i, -i\}$ forms a group under complex number multiplication. Firstly create table

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

  So we can see that it is closed. Complex number multiplication is associative[25]. Identity element: 1. Inverses: $1^{-1} = 1$, $(-1)^{-1} = -1$, $i^{-1} = -i$ and $(-i)^{-1} = i$.

- A group is commutative or Abelian if $a \circ b = b \circ a$ for all a and b in G. If you have a Latin Square for the group you can see if it is Abelian by seeing if it symmetrical along the leading diagonal.

- The *order of a group* is the number of elements the group contains. If a group contains an infinite number of elements it is said to be of *infinite order*.

[25] Do be careful what you can assume here! In the question it should tell you what you can assume. Beware of assuming anything that you are not told in the question. To assume makes an ass out of u and me!

- The *order of an element* a of G is the *smallest* n such that $a^n = e$. If no such n exists then the element is said to have *infinite order*. A group is *cyclic* if every element of a group can be generated by powers of a single element.

- A *subgroup* (of a group) is any non-empty subset of G which also forms a group under the same binary operation $\circ$. (A subgroup includes the subset containing just e and the subset G itself.) A *proper subgroup* is any subgroup with order not one or the same as the original group.

- A good way to find subgroups (beyond the cases where it is obvious) is to consider the powers of the elements of the original group; if you get back to e then the set of elements gone through will be a subgroup. For example in a group of order 16, if you take an element a and discover that $a^4 = e$ (i.e. the order of a is 4) then the set $\{e, a, a^2, a^3\}$ will form a subgroup.

- *Lagrange's theorem* states that the order of any subgroup must divide the order of the original group. For example a group of order 8 could potentially only have subgroups of order 1, 2, 4 or 8. It could therefore potentially only have proper subgroups of order 2 or 4. Some useful corollaries of Lagrange's Theorem include:
  - The order of an element *must* divide the order of the group.
  - A group of prime order *must* be cyclic.

- Two groups $(G, \circ)$ and $(H, \bullet)$ are isomorphic if there exists a one-to-one mapping between them which preserves their structure, i.e.

$$a \leftrightarrow x \text{ and } b \leftrightarrow y \qquad \Leftrightarrow \qquad a \circ b \leftrightarrow x \bullet y.$$

  A good way to show that groups are not isomorphic is to consider the orders of the elements of G and H: If they are different, then they *cannot* be isomorphic. In an exam you must make the mappings (something) $\leftrightarrow$ (something else) *very* clear; i.e. list them out!

- You need to know the structure of groups up to order 7. Groups of order 2, 3, 5 and 7 must be cyclic (prime order) and therefore every group of order p (say), must be isomorphic to every other group of order p. These groups are all isomorphic to $(\mathbb{Z}_p, +)$, the group of $\{0, 1, 2, p-1\}$ under addition mod p.

- There are two groups of order 4:

$(\mathbb{Z}_4, +)$

	e	a	a^2	a^3
e	e	a	a^2	a^3
a	a	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

and the Klein four-group

	e	a	b	ba
e	e	a	b	ba
a	a	e	ba	b
b	b	ba	e	a
ba	ba	b	a	e

  Whereas the cyclic group is generated by a single element a, the Klein four-group is generated by two elements, a and b with $a^2 = b^2 = e$ and $ab = ba$. In the Klein four-group every element is self inverse (i.e. has order 2).

- For groups of order 6 there are two fundamental types, the cyclic group isomorphic to $(\mathbb{Z}_6, +)$ and the dihedral group D_3 which represents the symmetries of the regular triangle under rotation and reflection. The group is generated by the rotation $\frac{2\pi}{3}$ (a) and reflection (b) with $a^3 = b^2 = e$ and $ab = ba^2$. The table is:

	e	a	a^2	b	ba	ba^2
e	e	a	a^2	b	ba	ba^2
a	a	a^2	e	ba^2	b	ba
a^2	a^2	e	a	ba	ba^2	b
b	b	ba	ba^2	e	a	a^2
ba	ba	ba^2	b	a^2	e	a
ba^2	ba^2	b	ba	a	a^2	e