OCR Further Pure 3 Module Revision Sheet

The FP3 exam is 1 hour 30 minutes long. You are allowed a graphics calculator.

Before you go into the exam make sure you are fully aware of the contents of the formula booklet you receive. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted... always an area prone to error.

J.M.S.

Differential Equations

First Order

- Given a first order differential equation, first see if it is separable *à la* C4.

  $$\frac{dy}{dx} = f(x)g(y) \qquad \Rightarrow \qquad \int \frac{dy}{g(y)} = \int f(x)\,dx.$$

  Remember to add the "$+c$" to one side only. If you leave the "$+c$" unevaluated then your answer represents the general solution; otherwise your answer is a particular solution. (This will hardly ever happen in FP3.) In these notes $P \equiv P(x)$ and $Q \equiv Q(x)$.

- Differential equations can sometimes be changed by a substitution into a more accessible form for solution. Usually this will be trivial, but sometimes replacing the differentiable bit will require the chain/product rules, and can be a little fiddly. For example find the general solution of

  $$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \text{ by the substitution } y = xu.$$

  Clearly the question is guiding us to get rid of all the y's in the original equation so we find

  $$\begin{aligned} \frac{d(xu)}{dx} &= \frac{x^2+(xu)^2}{2x(xu)}, \\ u + x\frac{du}{dx} &= \frac{1+u^2}{2u}, \\ x\frac{du}{dx} &= \frac{1-u^2}{2u}. \end{aligned}$$

  This is separable, and we obtain the solution $\frac{1}{1-u^2} = Ax$ for some arbitrary A. Eliminating u from this using $y = xu$ we find $y^2 = x^2 - kx$ for some constant k.

- If it is not separable and there is no substitution to try then try to re-arrange into the form

  $$\frac{dy}{dx} + P(x)y = Q(x).$$

  You will then need to multiply by an integrating factor (IF) which is defined as IF $= e^{\int P\,dx}$. Then we can[1] re-write the LHS in the form $d(\ldots)/dx$.

[1] By understanding that $\frac{d}{dx}(ye^{\int P\,dx}) = e^{\int P\,dx}\frac{dy}{dx} + Pe^{\int P\,dx}y$.

- Example solve the differential equation $x\frac{dy}{dx} + 2y = \frac{4}{x}$. We rearrange to get $\frac{dy}{dx} + \frac{2}{x}y = \frac{4}{x^2}$ the we work out the IF to be $e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$ so the equation becomes $x^2\frac{dy}{dx} + 2xy = 4$. We can now write the LHS to obtain

  $$\frac{d(x^2y)}{dx} = 4 \quad \Rightarrow \quad \int d(x^2y) = \int 4\,dx \quad \Rightarrow \quad x^2y = 4x + c \quad \Rightarrow \quad y = \frac{4}{x} + \frac{c}{x^2}.$$

  This represents the general solution of the differential equation. If we were told that $y = 6$ when $x = 1$ we would put this into the GS and find $c = 2$. This would give us a particular solution of $y = \frac{4}{x} + \frac{2}{x^2}$.

- *Homogeneous linear* differential equations are of the form $\frac{dy}{dx} + ay = 0$ or $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ for constant a and b.

  Non-homogeneous linear differential equations are of the form $\frac{dy}{dx} + ay = f(x)$ or $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$ for constant a and b.

- Linear (and *only* linear) differential equations can be approached by the *auxiliary equation method*: we try a solution of the form $y = Ae^{\lambda x}$ in the original equation made homogeneous and construct an auxiliary equation in λ. We can then find λ and this will give us a complementary function (CF). For example $\frac{dy}{dx} - 3y = 2x$ would be modified to $\frac{dy}{dx} - 3y = 0$. Then the AE would be $\lambda Ae^{\lambda x} - 3Ae^{\lambda x} = 0$ so $\lambda = 3$ so the CF would be $y = Ae^{3x}$. If the original equation was homogeneous then the CF *is* the GS of the equation, but if it is non-homogeneous then you need to find particular integral (PI) which we add to the CF to get the GS. GS = CF + PI.

- In the above example we need to find a particular integral (PI) based on the form of $f(x)$ which in this case is $2x$. You guess the type of it depending on the type of $f(x)$. Here is a table of logical trials:

$f(x)$	TRIAL
linear	$lx + m$
polynomial order n	$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$
trig functions involving sin or cos px	$l\sin px + m\cos px$
exponential involving e^{px}	ce^{px}

  So here we try a PI of $y = mx + n$. Putting it in we find $m - 3(mx + n) = 2x$ and equating coefficients we find $m = -\frac{2}{3}$ and $n = -\frac{2}{9}$. The general solution (GS) would then be

  $$\text{GS} = \text{CF} + \text{PI} \qquad \Rightarrow \qquad y = Ae^{3x} - \tfrac{2}{3}x - \tfrac{2}{9}.$$

Second Order

- Given a second order DE of the form

  $$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

  for constant a and b you construct the auxiliary equation[2] (AE) $\lambda^2 + a\lambda + b = 0$. The solution to the AE dictates the form of the complementary function (CF). There are three cases to consider depending on the type of solutions the AE has:

[2]This is not plucked out of thin air! The AE is obtained by trying a solution of the form $y = e^{\lambda x}$

	Type AE		CF
1.	Real and distinct roots, λ_1 and λ_2	$\Rightarrow$	$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
2.	Repeated real root, α.	$\Rightarrow$	$y = e^{\alpha x}(A + Bx)$
3.	Complex roots, $\alpha \pm i\beta$	$\Rightarrow$	$y = e^{\alpha x}(A\sin\beta x + B\cos\beta x)$

Notice that each CF has two arbitrary constants A and B. This makes sense, because we are effectively integrating twice and so would expect this[3].

- If you are dealing with a homogeneous equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ then, as with first order, the GS is just the CF and you are done (subject to boundary/initial conditions!).

- If you are dealing with a non-homogeneous equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$ then we need to find a particular integral (PI) with the same principles as above. For example solve the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin x + \cos x.$$

  This gives AE of $\lambda^2 - 3\lambda + 2 = 0$ so $\lambda = 1$ or $\lambda = 2$. Therefore the CF is $y = Ae^x + Be^{2x}$. Looking at $\sin x + \cos x$ we would clearly try $y = m\sin x + n\cos x$ as our PI. Putting this in we discover

$$\begin{aligned}(-m\sin x - n\cos x) - 3(m\cos x - n\sin x) + 2(m\sin x + n\cos x) &= \sin x + \cos x,\\ (-m + 3n + 2m)\sin x + (-n - 3m + 2n)\cos x &= \sin x + \cos x.\end{aligned}$$

  Equating coefficients of $\sin x$ and $\cos x$ we discover $m + 3n = 1$ and $n - 3m = 1$. This solves to $m = -\frac{1}{5}$ and $n = \frac{2}{5}$ giving us a GS of

$$y = Ae^x + Be^{2x} - \tfrac{1}{5}\sin x + \tfrac{2}{5}\sin x.$$

- The only *caveat* to the PI guesses is if the trial function for a PI is the same as one of CFs: you then multiply the trial function by x. For example find the general solution of

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}.$$

  The auxiliary equation gives $\lambda = 2$ or $\lambda = 3$. This gives a CF of $y = Ae^{2x} + Be^{3x}$. We would normally try a PI of $y = me^{2x}$, but we notice that e^{2x} is part of the CF, so we try a PI of $y = mxe^{2x}$ instead. Put this into the original differential equation and we find

$$\begin{aligned}(2me^{2x} + 2me^{2x} + 4mxe^{2x}) - 5(me^{2x} + 2mxe^{2x}) + 6mxe^{2x} &= e^{2x},\\ 2m + 2m + 4mx - 5m - 10mx + 6mx &= 1,\\ m &= -1.\end{aligned}$$

  Therefore the GS is $y = Ae^{2x} + Be^{3x} - xe^{2x}$.

- If you are given conditions to be satisfied by the system at the start, then you find the GS and then put in the information to find the value of the arbitrary constants.

[3] The number of arbitrary constants should always match the order of the equation.

Vectors

- Recall that a line through position vector $\mathbf{a}$ and with direction $\mathbf{d}$ is, in vector form, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$. Recall also that the line through position vectors $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$ or $\mathbf{r} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b})$ or equivalent. This is because when I subtract two position vectors $(\mathbf{b} - \mathbf{a})$ it yields the translation vector that travels from $\mathbf{a}$ to $\mathbf{b}$. For example find the line that passes through $(1, 3, 4)$ and $(3, 1, 8)$: This gives $\mathbf{r} = \begin{pmatrix}1\\3\\4\end{pmatrix} + \lambda \begin{pmatrix}2\\-2\\4\end{pmatrix}$ which would then simplify to give $\mathbf{r} = \begin{pmatrix}1\\3\\4\end{pmatrix} + \mu \begin{pmatrix}1\\-1\\2\end{pmatrix}$. Always look to simplify the direction vector if possible.

- A line can also be given in cartesian form
$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}.$$
To convert cartesian to vector form place each of the three elements equal to λ and 'unwrap'. For example $\frac{x-3}{2} = \frac{y+7}{2} = \frac{2-z}{3}$: so
$$\frac{x-3}{2} = \lambda \qquad \frac{y+7}{2} = \lambda \qquad \frac{2-z}{3} = \lambda.$$
So $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3+2\lambda\\-7+2\lambda\\2-3\lambda\end{pmatrix}$, therefore $\mathbf{r} = \begin{pmatrix}3\\-7\\2\end{pmatrix} + \lambda \begin{pmatrix}2\\2\\-3\end{pmatrix}$. To convert the other way should be trivial; unwind the above (practice for yourself).

- A plane in 3D can be given by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{p} + \mu\mathbf{q}$. You can think of this as a point $\mathbf{a}$ being stretched along the line with direction $\mathbf{p}$ and then that line being stretched along in direction $\mathbf{q}$ to form a plane. This is the least helpful form for the plane (IMHO). [If I saw this I would cross $\mathbf{p}$ and $\mathbf{q}$ to get $\mathbf{n}$ and then use the form $ax + by + cz = k$: see below.]

- A plane in 3D through point $\mathbf{a}$ and normal $\mathbf{n}$ can be given by $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$ where $\mathbf{r}$ represents all the points on the plane[4]. This is because if the dot product is zero then $(\mathbf{r} - \mathbf{a})$ and $\mathbf{n}$ must be at right angles and $(\mathbf{r} - \mathbf{a})$ is the vector that travels from $\mathbf{a}$ to any point $\mathbf{r}$. A nice sketch in the textbook P250. This can then be written $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$ so $\begin{pmatrix}x\\y\\z\end{pmatrix} . \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} = n_1x + n_2y + n_3z = \text{constant}$. Therefore...

- ...a plane can most usefully be in the form $ax + by + cz = k$ where $\begin{pmatrix}a\\b\\c\end{pmatrix}$ is the normal vector and k is some constant for *that* plane. As k varies it will produce parallel planes, like pages in a book.

- To find the intersection of a line and a plane is easy. Best done by example, find intersection of $\mathbf{r} = \begin{pmatrix}2\\1\\3\end{pmatrix} + t \begin{pmatrix}-1\\1\\3\end{pmatrix}$ and $2x + y - 3z = -24$. From the line we know that $x = 2 - t$ etc. so we place all of these into the plane and solve for t, so $2(2 - t) + (1 + t) - 3(3 + 3t) = -24$ which solves to give $t = 2$. Place this back into the line and the point is $(0, 3, 9)$.

[4] $\mathbf{r} = (x, y, z)$

- You also need to be able to determine whether a line lies *in* a plane or parallel to it. If you try to find where the line crosses the plane (like above) you will either boil your equation for λ down to a consistency ($1 = 1$) in which case the line lies in the plane, or an inconsistency ($0 = 1$) in which case the line is parallel to the plane.

- To find the shortest distance between a line and a point, do a sketch of the line and the point, and construct the triangle between the point away from the line (P), the point **a** on the line (A) and the point which is closest to the point (F). We want the length PF in the right angled triangle APF. The angle $P\widehat{A}F$ can be found by dotting the direction vector of the line with the vector $\overrightarrow{AP}$ and we can work out the length AP by working out the magnitude of $\overrightarrow{AP}$. Then it's just sin in a right angled triangle to get length PF.

- To find shortest distance between a point and a plane is a trivial extension of the above. Construct the line through the point with direction vector normal to the plane. Find where this line crosses the plane and then find the distance between these two points. [Quick reminder: The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$
or, equivalently, the distance between points with position vectors **a** and **b** is the magnitude of $(\mathbf{b} - \mathbf{a})$.]

- Must be able to find the intersection of two planes. For example find intersection of $x + 2y - 2z = 2$ and $2x + 3y - 7z = 1$: Up to you which variable you would like to eliminate, but I'd do twice the first minus the second. This gives $y + 3z = 3$. Let $z = t$ (you'd be silly to let $y = t$, try it to see why!) and we find $y = 3 - 3t$ and then $x = 2 - 2y + 2z = 2 - 2(3 - 3t) + 2(t) = -4 + 8t$. Therefore
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 + 8t \\ 3 - 3t \\ t \end{pmatrix} \quad \text{so} \quad \mathbf{r} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix}.$$
You could also have solved this by crossing the normals of the planes to discover the direction vector of the line of intersection and then find any point where the planes cross.

- The angle between two planes is the same as the angle between their normals. The angle between a plane and a line is $\frac{\pi}{2}$ minus the angle between the line and the plane's normal.

- The vector product of two vectors is defined
$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \widehat{\mathbf{n}}, \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} bn - cm \\ cl - an \\ am - bl \end{pmatrix}.$$
In practice you very rarely need the $\sin\theta$ bit of this definition (any angles you need are always more easily accessible by the dot product). It is most useful in the way it constructs a vector perpendicular to both original vectors.

- For example: Find the equation of the plane through $A(1, 2, -1)$,$B(2, 3, 4)$ and $C(-2, 0, 1)$. We construct the normal vector to the plane by crossing $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
$$\overrightarrow{AB} \times \overrightarrow{AC} = \left(\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right) \times \left(\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ -17 \\ 1 \end{pmatrix}.$$
Therefore the plane is $12x - 17y + z = \text{const}$. We find the constant by taking your favourite of A, B or C and plugging it in[5]. So $12x - 17y + z = -23$.

[5] A nice check is to put more than one point in and check that you get the same constant each time.

- The shortest distance between the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ is given by

$$d = \frac{\mid (\mathbf{a}_2 - \mathbf{a}_1).(\mathbf{d}_1 \times \mathbf{d}_2) \mid}{\mid \mathbf{d}_1 \times \mathbf{d}_2 \mid}.$$

Complex Numbers

- Recall that the complex number $z = a + ib$ has modulus $\mid z \mid \equiv r = \sqrt{a^2 + b^2}$ and argument $\arg(z) = \theta = \tan^{-1}\frac{b}{a}$, where argument is measured anti-clockwise from the positive real axis. Also recall that arguments can be of either convention $0 \leqslant \arg(z) < 2\pi$ or $-\pi < \arg(z) \leqslant \pi$. This is purely arbitrary and will be clear from the question what they want. [In my mind arguments are such that $-\infty < \arg(z) < \infty$ and they can keep twirling round, but OCR forces each complex number to have a unique argument.]

- By considering a right angled triangle like the one at the top of P295 we discover $z = a + ib = r(\cos\theta + i\sin\theta)$. This is the *polar form* of a complex number.

  It can also be shown[6] that $r(\cos\theta + i\sin\theta) \equiv re^{i\theta}$. This is the *exponential form* of a complex number and is unbelievably useful![7]

- Properties of complex number multiplication and division become immediately apparent:

$$re^{i\alpha} \times \rho e^{i\beta} = (r\rho)e^{i(\alpha+\beta)} \qquad \text{When multiplying, add arguments and multiply moduli,}$$
$$\frac{re^{i\alpha}}{\rho e^{i\beta}} = \left(\frac{r}{\rho}\right)e^{i(\alpha-\beta)} \qquad \text{When dividing, subtract arguments and divide moduli.}$$

  So given a complex number w, if I were to multiply w by the complex number $2e^{\frac{i\pi}{3}}$ (say) then the result would be the complex number with twice the length/modulus of w and rotated $\frac{\pi}{3}$ anti-clockwise from the original w. This is called a spiral-enlargement.

- De Moivre's Theorem[8] states that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad \text{or} \qquad (e^{i\theta})^n = e^{in\theta}.$$

  For integer n this can be proven by induction. The reason for the truth of De Moivre's theorem should be obvious from the above two properties for multiplication and division. To raise complex number w to the power four (say) the modulus would be raised to the power four, but the argument would be made four times bigger... which is what De Moivre says.

  We notice the special case

$$(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta \qquad \text{or} \qquad (e^{i\theta})^{-1} = e^{-i\theta}$$

  which comes up a lot; the inverse of a complex number with unit modulus is its complex conjugate.

- You can use De Moivre's theorem to derive certain trigonometric results. You need

$$\cos n\theta = \mathrm{Re}((\cos\theta + i\sin\theta)^n) \qquad \text{and} \qquad \sin n\theta = \mathrm{Im}((\cos\theta + i\sin\theta)^n).$$

  You can then use the standard relationship $\sin^2\theta + \cos^2\theta = 1$ to manipulate any raw results. Your algebra needs to be top-notch here; any slip at the start of a question can

[6] By mucking about with $\cos\theta + i\sin\theta = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots) + i(x - \frac{x^3}{3!} + \dots) = \text{lots of working} = e^{i\theta}$.

[7] It should be noted that an alternative notation for e^x at a higher level is $\exp(x)$. This is because the power on the e can sometimes become quite complicated.

[8] Abraham de Moivre, an 18th century statistician and consultant to gamblers. French...

cost you dearly on a multi-parter. For example express $\cos 5\theta$ in terms of powers of $\cos\theta$. So

$$\begin{aligned}\cos 5\theta &= \mathrm{Re}[(\cos\theta + i\sin\theta)^5]\\ &= \mathrm{Re}[\cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5]\\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\ &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.\end{aligned}$$

- For problems involving $\tan n\theta$ you would consider the expansions of $\sin n\theta$ and $\cos n\theta$ and use $\tan n\theta \equiv \dfrac{\sin n\theta}{\cos n\theta}$. For example

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{\mathrm{Im}[(\cos\theta + i\sin\theta)^3]}{\mathrm{Re}[(\cos\theta + i\sin\theta)^3]} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}.$$

You could then divide both numerator and denominator by $\cos^3\theta$ to obtain the 'nicer'

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

- Given any complex number of unit modulus $z = e^{i\theta} = \cos\theta + i\sin\theta$ it can easily be shown that

$$\frac{1}{2}\left(z + \frac{1}{z}\right) = \cos\theta \qquad \text{and} \qquad \frac{1}{2i}\left(z - \frac{1}{z}\right) = \sin\theta.$$

Similarly we can derive (by using De Moivre on the unit complex number; $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$) the very useful relations

$$\frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = \cos n\theta \qquad \text{and} \qquad \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right) = \sin n\theta.$$

So whereas we can use De Moivre to find multiple angle expressions (such as $\sin 6\theta$) in terms of powers of sin and cos, we can use the above to write powers of sin and cos (such as $\sin^7\theta$) in terms of multiple angles. For example express $\cos^6\theta$ as a sum of multiple angles of $\cos\theta$.

$$\begin{aligned}\cos^6\theta &= \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^6,\\ &= \frac{1}{64}\left(z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}\right),\\ &= \frac{1}{64}\left(\left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20\right),\\ &= \frac{1}{64}\left(2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20\right),\\ &= \frac{1}{32}\left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right).\end{aligned}$$

- You could then use the above to help you with integrals:

$$\begin{aligned}\int \cos^6\theta\, d\theta &= \frac{1}{32}\int (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\ d\theta,\\ &= \tfrac{1}{32}(\tfrac{1}{6}\sin 6\theta + \tfrac{3}{2}\sin 4\theta + \tfrac{15}{2}\sin 2\theta + 10\theta + c),\\ &= \frac{\sin 6\theta}{192} + \frac{3\sin 4\theta}{64} + \frac{15\sin 2\theta}{64} + \frac{5}{16} + c'.\end{aligned}$$

- You need to be able (using the ideas of roots of unity) to solve any equation of the form $z^n = a + ib$ (i.e. to get all n solutions to this equation). To get the *primary solution* you convert the $a + ib$ into the form $Re^{i\theta}$ and then obtain the first solution by raising both sides to the power $\frac{1}{n}$. So

$$\begin{aligned} z^n &= a + ib \\ z^n &= Re^{i\theta} \\ z &= (Re^{i\theta})^{1/n} \\ z &= \sqrt[n]{R}\, e^{\frac{i\theta}{n}}. \end{aligned}$$

  This is the primary solution and then you find the others by using the fact they are all evenly spaced around the circle of radius $\sqrt[n]{R}$, like spokes on a bike. So you keep adding $\frac{2\pi}{n}$ on to the argument of $\sqrt[n]{R}e^{\frac{i\theta}{n}}$, $n - 1$ times to get all n solutions to the equation.

- For example solve $z^4 = -16$. We rewrite in the form $z^4 = 16e^{i\pi}$. Therefore the primary solution is $z = 2e^{\frac{i\pi}{4}}$. Adding on $2\pi/4 = \pi/2$ to the arguments we find the four solutions

$$z = 2e^{\frac{i\pi}{4}}, 2e^{\frac{i3\pi}{4}}, 2e^{\frac{i5\pi}{4}}, 2e^{\frac{i7\pi}{4}}.$$

  If need be you could then convert these back to $a + ib$ form and get:

$$z = (\sqrt{2} + i\sqrt{2}), (-\sqrt{2} + i\sqrt{2}), (-\sqrt{2} - i\sqrt{2}), (\sqrt{2} - i\sqrt{2}).$$

  By taking the roots in complex conjugate pairs you could then factorise $z^4 + 16$ into the product of two real quadratic factors as

$$\begin{aligned} z^4 + 16 &= (z - (\sqrt{2} + i\sqrt{2}))(z - (\sqrt{2} - i\sqrt{2}))(z - (-\sqrt{2} + i\sqrt{2}))(z - (-\sqrt{2} - i\sqrt{2})) \\ &= (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4). \end{aligned}$$

Groups

- A group $(G, \circ)$ is a non-empty set G with a binary operation $\circ$ which
  - is **closed**, (for every a and b in G, $a \circ b$ also lies in G),
  - is **associative**, (for every a, b and c in G, $(a \circ b) \circ c = a \circ (b \circ c)$),
  - has a unique **identity** element, (an element e such that $e \circ a = a \circ e = a$ for all a in G),
  - every element has its own **inverse**, (for every a in G there exists a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = e$).

  A group can be represented as a Latin square. For example

	e	a	b
e	$e \circ e$	$e \circ a$	$e \circ b$
a	$a \circ e$	$a \circ a$	$a \circ b$
b	$b \circ e$	$b \circ a$	$b \circ b$

  Each row and column must contain every element of G once only. You can find the identity easily from this by looking for the row or column which is unchanged. Inverses are easy to find from a Latin Square; you merely look for which other element makes it the identity.

- If you are asked to show that something is a group in an exam you must tick off each of the above criteria one-by-one. For example show that the set $\{1, -1, i, -i\}$ forms a group under complex number multiplication. Firstly create table

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

  So we can see that it is closed. Complex number multiplication is associative[9]. Identity element: 1. Inverses: $1^{-1} = 1$, $(-1)^{-1} = -1$, $i^{-1} = -i$ and $(-i)^{-1} = i$.

- A group is commutative or Abelian if $a \circ b = b \circ a$ for all a and b in G. If you have a Latin Square for the group you can see if it is Abelian by seeing if it symmetrical along the leading diagonal.

- The *order of a group* is the number of elements the group contains. If a group contains an infinite number of elements it is said to be of *infinite order.*

- The *order of an element* a of G is the *smallest* n such that $a^n = e$. If no such n exists then the element is said to have *infinite order.* A group is *cyclic* if every element of a group can be generated by powers of a single element.

- A *subgroup* (of a group) is any non-empty subset of G which also forms a group under the same binary operation $\circ$. (A subgroup includes the subset containing just e and the subset G itself.) A *proper subgroup* is any subgroup with order not one or the same as the original group.

- A good way to find subgroups (beyond the cases where it is obvious) is to consider the powers of the elements of the original group; if you get back to e then the set of elements gone through will be a subgroup. For example in a group of order 16, if you take an element a and discover that $a^4 = e$ (i.e. the order of a is 4) then the set $\{e, a, a^2, a^3\}$ will form a subgroup.

- *Lagrange's theorem* states that the order of any subgroup must divide the order of the original group. For example a group of order 8 could potentially only have subgroups of order 1, 2, 4 or 8. It could therefore potentially only have proper subgroups of order 2 or 4. Some useful corollaries of Lagrange's Theorem include:
  - The order of an element *must* divide the order of the group.
  - A group of prime order *must* be cyclic.

- Two groups $(G, \circ)$ and $(H, \bullet)$ are isomorphic if there exists a one-to-one mapping between them which preserves their structure, i.e.

$$a \leftrightarrow x \text{ and } b \leftrightarrow y \qquad \Leftrightarrow \qquad a \circ b \leftrightarrow x \bullet y.$$

  A good way to show that groups are not isomorphic is to consider the orders of the elements of G and H: If they are different, then they *cannot* be isomorphic. In an exam you must make the mappings (something) $\leftrightarrow$ (something else) *very* clear; i.e. list them out!

[9] Do be careful what you can assume here! In the question it should tell you what you can assume. Beware of assuming anything that you are not told in the question. To assume makes an ass out of u and me!

- You need to know the structure of groups up to order 7. Groups of order 2, 3, 5 and 7 must be cyclic (prime order) and therefore every group of order p (say), must be isomorphic to every other group of order p. These groups are all isomorphic to $(\mathbb{Z}_p, +)$, the group of $\{0, 1, 2, p-1\}$ under addition mod p.

- There are two groups of order 4:

$(\mathbb{Z}_4, +)$

	e	a	a^2	a^3
e	e	a	a^2	a^3
a	a	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

and the Klein four-group

	e	a	b	ba
e	e	a	b	ba
a	a	e	ba	b
b	b	ba	e	a
ba	ba	b	a	e

Whereas the cyclic group is generated by a single element a, the Klein four-group is generated by two elements, a and b with $a^2 = b^2 = e$ and $ab = ba$. In the Klein four-group every element is self inverse (i.e. has order 2).

- For groups of order 6 there are two fundamental types, the cyclic group isomorphic to $(\mathbb{Z}_6, +)$ and the dihedral group D_3 which represents the symmetries of the regular triangle under rotation and reflection. The group is generated by the rotation $\frac{2\pi}{3}$ (a) and reflection (b) with $a^3 = b^2 = e$ and $ab = ba^2$. The table is:

	e	a	a^2	b	ba	ba^2
e	e	a	a^2	b	ba	ba^2
a	a	a^2	e	ba^2	b	ba
a^2	a^2	e	a	ba	ba^2	b
b	b	ba	ba^2	e	a	a^2
ba	ba	ba^2	b	a^2	e	a
ba^2	ba^2	b	ba	a	a^2	e