OCR Core 4 Module Revision Sheet

The C4 exam is 1 hour 30 minutes long. You are allowed a graphics calculator.

Before you go into the exam make sure you are fully aware of the contents of the formula booklet you receive. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted. . . always an area prone to error.

J .M .S .

Algebra

- Review binomial expansion from C2 for $(x+y)^n$ for positive integer n. Notice that it is valid for *any* x and y and that the expansion has $n+1$ terms.
- The general binomial expansion is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

and is valid for any n (fractional or negative) but $-1 < x < 1$ (i.e. $|x| < 1$). Notice also it must start with a 1 in the brackets. For example expand $(4-x)^{-1/2}$.

$$\begin{aligned}(4-x)^{-1/2} &= \left(4\left(1-\frac{x}{4}\right)\right)^{-1/2}\\ &= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}\\ &= \frac{1}{2}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{x}{4}\right)^2+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{x}{4}\right)^3+\cdots\right]\\ &= \frac{1}{2}\left[1+\frac{x}{8}+\frac{3x^2}{128}+\frac{15x^3}{3072}+\cdots\right] = \frac{1}{2}+\frac{x}{16}+\frac{3x^2}{256}+\frac{15x^3}{6144}+\cdots.\end{aligned}$$

It is only valid for $|x/4| < 1 \quad \Rightarrow \quad |x| < 4$.

- Another example: Find first 3 terms in the expansion for $\dfrac{(3+x)^2}{1+\frac{x}{2}}$.

$$(3+x)^2\left(1+\frac{x}{2}\right)^{-1} = (9+6x+x^2)\left(1-\frac{x}{2}+\frac{x^2}{4}+\ldots\right) = 9+\frac{3}{2}x+\frac{1}{4}x^2+\ldots.$$

- You must be able to simplify algebraic fractions; best tactic is always to factorise and cancel:

$$\frac{4x^5-10x^4-6x^3}{12x^6-18x^5-12x^4} = \frac{2x^3(2x^2-5x-3)}{6x^4(2x^2-3x-2)} = \frac{2x^3(2x+1)(x-3)}{6x^4(x-2)(2x+1)} = \frac{x-3}{3x(x-2)}.$$

- You must be able to divide a polynomial $(p(x))$ by a divisor $(a(x))$, finding the quotient $(q(x))$ and remainder $(r(x))$. If there is no remainder then $a(x)$ is a factor of $p(x)$. It is always such that

$$\frac{p(x)}{a(x)} = q(x) + \frac{r(x)}{a(x)} \qquad \Rightarrow \qquad p(x) = a(x)q(x) + r(x).$$

The order of $q(x)$ is the order of $p(x)$ subtract the order of $a(x)$. The order of $r(x)$ is *at most* one less than $a(x)$. For example if you have a quintic (power 5 polynomial) divided by a quadratic you would expect

$$\frac{\text{quintic}}{\text{quadratic}} = \text{cubic} + \frac{\text{linear}}{\text{quadratic}},$$
$$\frac{\text{quintic}}{\text{quadratic}} = Ax^3 + Bx^2 + Cx + D + \frac{Ex+F}{\text{quadratic}}.$$

Of course it *may* turn out that $Ex + F$ is just a constant or zero (if the cubic divides the quintic).

- Division is most easily done step-by-step working *down* the powers of $p(x)$. For example divide $2x^4 - x^3 + 3x^2 - 7x + 1$ by $x^2 + 2x + 3$:

$$\begin{aligned}
2x^4 - x^3 + 3x^2 - 7x + 1 &= (x^2+2x+3)(\text{quadratic}) + (\text{remainder}) \\
&= (x^2+2x+3)(2x^2 + \dots) + (\text{remainder}) && x^4\checkmark \\
&= (x^2+2x+3)(2x^2 - 5x + \dots) + (\text{remainder}) && x^3\checkmark \\
&= (x^2+2x+3)(2x^2 - 5x + 7) + (\text{remainder}) && x^2\checkmark \\
&= (x^2+2x+3)(2x^2 - 5x + 7) - 6x + \text{const.} && x\checkmark \\
&= (x^2+2x+3)(2x^2 - 5x + 7) - 6x - 20 && \text{const.}\checkmark
\end{aligned}$$

  Therefore $q(x) = 2x^2 - 5x + 7$ and $r(x) = -6x - 20$.

- Partial fractions is effectively the reverse of combining together two algebraic fractions. For example

$$\frac{1}{x+1} + \frac{1}{x+2} \quad \begin{matrix} \longrightarrow & \text{Algebraic Fractions} & \longrightarrow \\ \longleftarrow & \text{Partial Fractions} & \longleftarrow \end{matrix} \quad \frac{2x+3}{(x+1)(x+2)}.$$

  You can use partial fractions provided the order of the top line is less than the order of the bottom line.

- If you simply have a product of linear factors on the bottom line then you split out into that many terms with constants placed on top (usually denoted by A, B, C, etc.). Place this equal (in an identity "$\equiv$") to the original expression and then multiply through to get rid of the denominators. For example:

$$\begin{aligned}
\frac{7x-1}{(2x+1)(x-1)} &\equiv \frac{A}{2x+1} + \frac{B}{x-1} \\
\Rightarrow \qquad 7x - 1 &\equiv (x-1)A + (2x+1)B.
\end{aligned}$$

  Because this is an identity we can choose any value of x we fancy to help us discover A and B. In this case letting $x = 1$ is a good choice because one of the brackets become zero. Similarly $x = -\frac{1}{2}$ is another good choice[1]; we put these into the identity.

$$\begin{aligned}
x = 1 \quad &\Rightarrow \quad 7 - 1 \equiv 3B \quad \Rightarrow \quad \underline{B = 2}, \\
x = -\tfrac{1}{2} \quad &\Rightarrow \quad -\tfrac{7}{2} - 1 \equiv -\tfrac{3}{2}A \quad \Rightarrow \quad \underline{A = 3}.
\end{aligned}$$

  Therefore $\dfrac{7x-1}{(2x+1)(x-1)} \equiv \dfrac{3}{2x+1} + \dfrac{2}{x-1}$.

[1]If you don't spot (or run out of) clever values to use, fret not! Just put in some other numbers (usually something 'nice' like $x = 0$, $x = 1$ or $x = -1$) and solve the resulting equations. If your girlfriend's lucky number is 53 and your mistress's lucky number is 178 then feel free to use those if you wish, although I wouldn't advise it; choose simple numbers!

- If you have a repeated factor in the denominator then you deal with it as follows (notice the top line is a quadratic and the bottom a cubic, so partial fractions are fine):

$$\frac{5x^2-10x+1}{(x-3)(x-1)^2} \equiv \frac{A}{x-3}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$
$$\Rightarrow \quad 5x^2-10x+1 \equiv (x-1)^2A+(x-3)(x-1)B+(x-3)C.$$

Similarly, good values of x to choose are $x=1$ and $x=3$:

$$x=1 \quad \Rightarrow \quad 5-10+1 \equiv -2C \quad \Rightarrow \quad \underline{C=2},$$
$$x=3 \quad \Rightarrow \quad 45-30+1 \equiv 4A \quad \Rightarrow \quad \underline{A=4}.$$

This tells us that

$$5x^2-10x+1 \equiv 4(x-1)^2+(x-3)(x-1)B+2(x-3),$$

but it hasn't told us B. I would consider $x=0$, here, and sub in to discover

$$0-0+1 \equiv 4+3B-6 \Rightarrow \underline{B=1}.$$

Therefore $\frac{5x^2-10x+1}{(x-3)(x-1)^2} \equiv \frac{4}{x-3}+\frac{1}{x-1}+\frac{2}{(x-1)^2}$.

- Partial fractions are often very useful in evaluating integrals. If you see a quadratic in the bottom line of an integral, then one option your brain should turn to is "can this be split into partial fractions?". For example in this integral, it can be split and then integrated:

$$\int \frac{x+10}{x^2+5x+4}\,dx = \int \frac{x+10}{(x+1)(x+4)}\,dx$$
$$= \int \frac{3}{x+1} - \frac{2}{x+4}\,dx$$
$$= 3\ln(x+1) - 2\ln(x+4) + c.$$

Differentiation & Integration

- Know the contents of the formula booklet well. Very well! Lots of problems can be solved simply by looking at the table of differentials and integrals and knowing that integration 'undoes' a differentiation. Some questions get you to differentiate something and *then* get you to integrate something similar. *Always view the question as a whole!*

- When using radians we can differentiate the trigonometric functions. The results are as follows:

$$y=\sin x \qquad y=\cos x \qquad y=\tan x$$
$$\frac{dy}{dx}=\cos x, \qquad \frac{dy}{dx}=-\sin x, \qquad \frac{dy}{dx}=\sec^2 x.$$

One can derive the third result from the other two using the quotient rule and that $\tan x \equiv \frac{\sin x}{\cos x}$.

- You can also use these results along with the chain rule to differentiate functions like the following; $y=\sin(x^2+1)$ by letting $u=x^2+1$ and $y=(\tan x)^{10}$ by letting $u=\tan x$.

$$y=\sin(x^2+1) \qquad y=(\tan x)^{10}$$
$$\frac{dy}{dx}=2x\cos(x^2+1), \qquad \frac{dy}{dx}=10\sec^2 x(\tan x)^9.$$

- Integration by substitution is a way of integrating by replacing the variable given to you (usually x) and replacing it by another (usually u). These days the substitution you are to use is given to you in the exam, but practice will get you better at spotting what to substitute (usually the most complicated term in the integration or the denominator of a fraction). For example $\int x^3(x^4+1)^7\,dx$ we should use $u = x^4+1$.

$$\begin{aligned}
&\int x^3(x^4+1)^7\,dx && u = x^4+1\\
=&\int x^3u^7\,dx && \frac{du}{dx} = 4x^3\\
=&\int x^3u^7\,\frac{du}{4x^3} && \frac{du}{4x^3} = dx\\
=&\frac{1}{4}\int u^7\,du &&\\
=&\frac{u^8}{32}+c = \frac{(x^4+1)^8}{32}+c. &&
\end{aligned}$$

  We have effectively *"used and abused"* u to help us to get the answer. (Note: I have been *very* sloppy in the above integration because I have mixed my x and u variables; you shouldn't really do this, but it makes the process of conversion clearer.)

- When dealing with definite integrals we need to also convert the limits of the integration and there is no need to convert back to x at the end since all definite integrals are merely numbers. For example

$$\begin{aligned}
&\int_3^4 2x\sqrt{x^2-4}\,dx && u = x^2-4 && x=3 \Rightarrow u=5\\
=&\int_5^{12} 2xu^{1/2}\,\frac{du}{2x} && \frac{du}{dx} = 2x && x=4 \Rightarrow u=12\\
=&\int_5^{12} u^{1/2}\,du && \frac{du}{2x} = dx &&\\
=&\left[\frac{2}{3}u^{3/2}\right]_5^{12} && &&\\
=&\,20.3 \text{ (3sf)}. && &&
\end{aligned}$$

- Know the result $\int e^{ax}\,dx = \frac{1}{a}e^{ax}+c$.

- We know that if $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$. Therefore by reversal we find

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)}+c.$$

  For example[2]

$$\int x^3e^{x^4}\,dx = \frac{1}{4}\int 4x^3e^{x^4}\,dx = \frac{1}{4}e^{x^4}+c.$$

- Know that $\int \frac{1}{x}\,dx = \ln x + c$.

- We know (by the chain rule) that if $y = \ln(f(x))$ then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$. Therefore by reversal we find

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+c.$$

[2]This could also have been evaluated (more slowly) by a substitution of $u = x^4$ which would then have reduced to $\int x^3e^{x^4}\,dx = \frac{1}{4}\int e^u\,du = \frac{1}{4}e^{x^4}+c$.

Be on the lookout for expressions where the top line is almost the derivative of the bottom line. For example[3]

$$\int \frac{x^3}{x^4+1}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4+1}\,dx = \frac{1}{4}\ln|x^4+1| + c.$$

- Know the results

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + c \qquad \text{and} \qquad \int \sin ax\,dx = -\frac{1}{a}\cos ax + c.$$

- Always be on the look out for integrals involving a mixture of trigonometric functions. These are usually handled by means of a substitution. For example $\int \cos x(\sin x)^7\,dx$ is best handled by $u = \sin x$ to give $\frac{1}{8}(\sin x)^8 + c$.

- Also know the useful results (all derived from reverse chain rule)

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c \qquad \text{and} \qquad \int f'(x)\sin f(x)\,dx = -\cos f(x) + c.$$

  For example $\int x^3\cos(x^4)\,dx = \frac{1}{4}\sin(x^4) + c$.

- When an integral is made up of two 'bits' then we can sometimes use *integration by parts*. It states

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

  So you will need to decide which 'bit' of the integral you will need to differentiate and which 'part' to integrate. For example in $\int x\sin x\,dx$ it is quite clear that we will need to differentiate the x 'part' and integrate the $\sin x$ 'part'.

$$\begin{aligned}\int x\sin x\,dx &= -x\cos x - \int -\cos x\,dx\\ &= -x\cos x + \sin x + c.\end{aligned}$$

- Another example (this time a definite integral)

$$\begin{aligned}\int_0^2 xe^{2x}\,dx &= \left[\tfrac{1}{2}xe^{2x}\right]_0^2 - \int_0^2 \tfrac{1}{2}e^{2x}\,dx\\ &= \left[\tfrac{1}{2}xe^{2x}\right]_0^2 - \left[\tfrac{1}{4}e^{2x}\right]_0^2\\ &= \left(e^4 - 0\right) - \left(\tfrac{e^4}{4} - \tfrac{1}{4}\right) = \tfrac{3e^4}{4} + \tfrac{1}{4}.\end{aligned}$$

- Initially $\int \ln x\,dx$ looks nothing like it has anything to do with integration by parts because it only has one 'part'. However if we write $\ln x$ as $1 \times \ln x$ we can integrate the 1 and differentiate the $\ln x$:

$$\int \ln x\,dx = \int 1\ln x\,dx = x\ln x - \int x.\tfrac{1}{x}\,dx = x\ln x - x + c.$$

  This principle can be extended to integrals of the type $\int x^n\ln x\,dx$:

$$\int x^n\ln x\,dx = \tfrac{x^{n+1}}{n+1}\ln x - \int \tfrac{x^{n+1}}{n+1}.\tfrac{1}{x}\,dx = \tfrac{x^{n+1}}{n+1}\ln x - \tfrac{x^{n+1}}{(n+1)^2} + c.$$

[3] Again, this could also have been evaluated by the substitution $u = x^4 + 1$.

- Very occasionally you will need to integrate by parts *twice* to get the final answer. This will almost always be of the form $\int kx^2(\text{something})\,dx$. For example find $\int x^2e^{2x}\,dx$:

$$\begin{aligned}\int x^2e^{2x}\,dx &= \frac{x^2}{2}e^{2x} - \left(\int xe^{2x}\,dx\right)\\ &= \frac{x^2}{2}e^{2x} - \left(\frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx\right)\\ &= \frac{x^2}{2}e^{2x} - \left(\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right) + c\\ &= \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + c.\end{aligned}$$

- For the cases of $\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$ we need to recall two forms of the double angle formula for $\cos 2x$: Namely $\cos 2x = 1 - 2\sin^2 x$ (for $\int \sin^2 x\,dx$) and $\cos 2x = 2\cos^2 x - 1$ (for $\int \cos^2 x\,dx$). Re-arranging them both we find:

$$\begin{aligned}\int \sin^2 x\,dx &= \int (\tfrac{1}{2} - \tfrac{1}{2}\cos 2x)\,dx = \tfrac{x}{2} - \tfrac{1}{4}\sin 2x + c,\\ \int \cos^2 x\,dx &= \int (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\,dx = \tfrac{x}{2} + \tfrac{1}{4}\sin 2x + c.\end{aligned}$$

  Learn the technique rather than the result!

Implicit Functions

- Given a function in the form $y = f(x)$ we can differentiate it. Implicit differentiation allows us to differentiate a function without making y the subject first. It uses the chain rule that

$$\frac{d\,f(y)}{dx} = \frac{d\,f(y)}{dy} \times \frac{dy}{dx}.$$

  So all you do is differentiate the y bits with respect to y and then multiply by $\frac{dy}{dx}$. For example differentiate $y^4 + x^4 = \sin y$ with respect to x. This gives

$$4y^3\frac{dy}{dx} + 4x^3 = \cos y\frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{4x^3}{\cos y - 4y^3}.$$

  You must be on the lookout for products in terms of x and y; for example $2xy = e^{2y}$ would differentiate to

$$2x\frac{dy}{dx} + 2y = 2e^{2y}\frac{dy}{dx} \quad \text{so} \quad \frac{dy}{dx} = \frac{2y}{2e^{2y} - 2x} = \frac{y}{e^{2y} - x}.$$

- Another example; find all the stationary points on the curve $x^2 + y^2 + xy = 3$. Differentiating w.r.t. x we find

$$2x + 2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{2x+y}{2y+x}.$$

  Stationary points are where $\frac{dy}{dx} = 0$ so solve

$$0 = -\frac{2x+y}{2y+x} \qquad \Rightarrow \qquad y = -2x.$$

  Substituting this *back into the original equation* we find

$$x^2 + (-2x)^2 + x(-2x) = 3 \qquad \Rightarrow \qquad x = \pm 1 \qquad \Rightarrow \qquad \text{Points are } (1,-2) \text{ and } (-1,2).$$

- If you discover $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$ and are asked to find where the tangents to a curve are parallel to the y-axis (i.e. vertical) then you need to solve where the bottom line is zero, i.e. solve $g(x,y) = 0$.

Parametric Equations

- A parametric equation is one where

$$x = f(\text{some parameter}) \quad \text{and} \quad y = g(\text{some parameter}).$$

  The parameter in a set of parametric equations can be any letter, but usually either t or θ. As the parameter varies it sketches out a curve. If no restriction is given, assume the parameter varies $-\infty < t < \infty$. However the parameter can be restricted in any way, defined by an inequality on the parameter. Standard examples: $0 \leqslant \theta < 2\pi$ or $-\pi < \theta \leqslant \pi$.

- You must be able to convert a parametric curve to Cartesian form. Sometimes this is just obvious; isolate t from one of the equations and put into the other. For example

$$\begin{matrix} x = 2t \\ y = \frac{t}{t+1} \end{matrix} \quad \Rightarrow \quad t = \frac{x}{2} \quad \Rightarrow \quad y = \frac{\frac{x}{2}}{\frac{x}{2}+1} = \frac{x}{x+2}.$$

  If one of x or y involves a "sin" and the other involves a "cos" then use $\sin^2 x + \cos^2 x = 1$:

$$\begin{matrix} x = 3\cos\theta \\ y = \sin\theta + 4 \end{matrix} \quad \Rightarrow \quad \begin{matrix} \cos^2\theta = \left(\frac{x}{3}\right)^2 \\ \sin^2\theta = (y-4)^2 \end{matrix} \quad \Rightarrow \quad \frac{x^2}{9} + (y-4)^2 = 1.$$

- To find where a line intersects a parametric curve, place the parameters (in terms of t) into the line and solve for t. For example find the points of intersection of

$$\begin{matrix} x = 2t^2 + 1 \\ y = \frac{1}{t} \end{matrix} \qquad \text{and the line} \qquad x + 4y = 7.$$

  Replace the x and y in the line by $2t^2 + 1$ and $\frac{1}{t}$ respectively. Therefore

$$x + 4y = 7, \qquad \Rightarrow \qquad (2t^2+1) + 4\left(\frac{1}{t}\right) = 7, \qquad \Rightarrow \qquad t^3 - 3t + 2 = 0.$$

  This cubic factorises to $(t-1)^2(t+2) = 0$ which gives $t = 1$ or $t = -2$ as solutions. Plugging these back into the original parametric equation we discover the two points $(3,1)$ and $(9, -\frac{1}{2})$. [It is worth noting that the squared factor $(t-1)^2$ in the cubic implies the the line is a *tangent* to the curve at the point $(3,1)$.]

- To differentiate a parametric curve

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

  For stationary points you still equate $\frac{dy}{dx} = 0$ and solve. All other properties you are used to for normals and tangents still work.

For example find the equation of the normal to $x = 2t^3$, $y = \frac{1}{t}$ at the point $(16, \frac{1}{2})$. Firstly we need to discover the value of the parameter at the stated point: $y = \frac{1}{t} = \frac{1}{2}$ implies $t = 2$. Next differentiate and put in $t = 2$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-t^{-2}}{6t^2} = -\frac{1}{6t^4}.$$

$$\text{When } t = 2,\ \frac{dy}{dx} = -\frac{1}{96}.$$

Therefore the gradient of the normal is 96. Thus, $y - \frac{1}{2} = 96(x - 16)$ which 'simplifies' to $192x - 2y - 3071 = 0$.

- In harder examples questions will leave the parameter unevaluated; either leaving it as t or setting $t = p$. For example, find the equation of the tangent to the curve $x = 2t$, $y = \frac{1}{t^2}$ where $t = p$. When $t = p$, the point becomes $(2p, \frac{1}{p^2})$. Differentiating we find

$$\frac{dy}{dx} = \frac{-2t^{-3}}{2} = -\frac{1}{t^3}.$$

Therefore the gradient of the tangent when $t = p$ is $-\frac{1}{p^3}$. Therefore the tangent is

$$y - \frac{1}{p^2} = -\frac{1}{p^3}(x - 2p) \qquad \Rightarrow \qquad x + p^3 y = 3p.$$

The question could further be extended to find the area of the triangle formed by the points where the tangent crosses the x-axis and y-axis and the origin. The tangent $(x + p^3 y = 3p)$ crosses the x-axis when $y = 0$ which gives $x = 3p$. The tangent crosses the y-axis when $x = 0$ which gives $y = \frac{3}{p^2}$. So the three vertices of the triangle are at $(0, 0)$, $(0, \frac{3}{p^2})$ and $(3p, 0)$. The area of the triangle is therefore

$$\text{Area} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2} \times 3p \times \tfrac{3}{p^2} = \tfrac{3}{2p}.$$

Differential Equations

- If you are told that ("something") is proportional to ("something else") then we write ("something") $\propto$ ("something else"). This implies that

$$(\text{“something”}) = \pm k(\text{“something else”})$$

for some *constant* k. k can then be determined by putting in one pair of values (x, y) into the equation. If you read that something is decreasing then use $-k$, if it is increasing then use $+k$. The expression "varies as" also implies proportionality between two quantities.

- The words "rate of change of (something)" $\Rightarrow \dfrac{d(\text{something})}{dt}$. Also the word "initially" $\Rightarrow$ $t = 0$. Also be on the lookout for phrases such as "where t is measured from now" implying $t = 0$ now.

- In simple cases you need to be able to construct a differential equation of a situation. For example: The number of people infected with bird flu (N) is growing at a rate proportional to the square of the number of people infected:

$$\frac{dN}{dt} \propto N^2 \qquad \Rightarrow \qquad \frac{dN}{dt} = +kN^2.$$

- The notation dy/dx lets us believe the it is a normal fraction. Although this is not the case we can manipulate it like a fraction in a differential equation. You must move the variables to different sides of the equation and integrate (separation of variables). Only add the ever-present "$+c$" to one side. For example solve

$$\frac{dy}{dx} = y^2 \cos x \quad \Rightarrow \quad \int \frac{1}{y^2}\,dy = \int \cos x\,dx \quad \Rightarrow \quad y = -\frac{1}{\sin x + c}.$$

$$\frac{dN}{dt} = +kN^2 \quad \Rightarrow \quad \int \frac{1}{N^2}\,dN = \int k\,dt \quad \Rightarrow \quad N = \frac{-1}{kt + c}.$$

  In the second example above you will notice that there are two constants; the constant of proportionality and the arbitrary integration constant. This means you will need to be given two pieces of data (t_1, N_1) and (t_2, N_2) to figure them both out.

- A final example involving partial fractions:

$$\begin{aligned}
(3P+1)\frac{dP}{dt} &= kt(P-1)(P+3) \\
\int \tfrac{3P+1}{(P-1)(P+3)}\,dP &= \int kt\,dt \\
\int \tfrac{1}{P-1} + \tfrac{2}{P+3}\,dP &= \tfrac{kt^2}{2} + c \\
\ln(P-1) + 2\ln(P+3) &= \tfrac{kt^2}{2} + c \\
\ln(P-1)(P+3)^2 &= \tfrac{kt^2}{2} + c.
\end{aligned}$$

- If the arbitrary constant is left unevaluated, then your solution represents the *general solution* of the differential equation. If you put a value in to work out its value then your solution is called the *particular solution*.

Vectors

- The vector $\begin{pmatrix}3\\4\end{pmatrix}$ can be written $3\mathbf{i} + 4\mathbf{j}$ and represents a vector going 3 right and 4 up. By Pythagoras' Theorem it can be shown that the magnitude of this vector is $\sqrt{3^2+4^2} = 5$ and by trigonometry the direction is $\tan^{-1}\frac{4}{3}$ above the horizontal.

- Two vectors are equal if their magnitudes and directions are the same. Two vectors are parallel if one is a scalar multiple of the other. For example show that $\begin{pmatrix}2\\3\end{pmatrix}$ is parallel to $\begin{pmatrix}3\\4.5\end{pmatrix}$; so show that $1.5 \times \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}3\\4.5\end{pmatrix}$.

- When multiplying a vector by a positive scalar it changes the length of the vector but not the direction. If the scalar is negative then it also reverses the direction of the vector. For example $3\begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}6\\-9\end{pmatrix}$.

- When adding vectors, you just add the x components and add the y components. For example $\begin{pmatrix}2\\-3\end{pmatrix} + \begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$.

- A unit vector is a vector with a magnitude 1. A unit vector in a given direction can be constructed by dividing a vector by its magnitude. For example the unit vector from $\begin{pmatrix}2\\3\end{pmatrix}$ is $\frac{1}{\sqrt{2^2+3^2}} \times \begin{pmatrix}2\\3\end{pmatrix} = \frac{1}{\sqrt{13}} \times \begin{pmatrix}2\\3\end{pmatrix}$.

- You must know the geometric interpretation of $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$. Also know that in general if you have position vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ then $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

- It cannot be stressed enough that <u>*subtraction*</u> is the most important operation with vectors. If you wish to travel *from* one point ($\mathbf{a}$) *to* another ($\mathbf{b}$) then we use subtraction: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

  I will repeat that! If you wish to travel *from* $\mathbf{a}$ *to* $\mathbf{b}$ then use subtraction: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

- If you wish to calculate a length in 3D space then you merely need to calculate the magnitude of the vector that travels between the two points (i.e. $|\mathbf{b} - \mathbf{a}|$).

- A line can be written in vector form. If you know a line goes through a point (a, b) and has the gradient m then its vector form is $\mathbf{r} = \begin{pmatrix} a \\ b \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix}$ where λ is a scalar that takes different values on different points on the line. The vector $\begin{pmatrix} 1 \\ m \end{pmatrix}$ can be re-written to make the components 'nicer'. For example $\begin{pmatrix} 1 \\ 2/3 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. (These vectors are not equal, but they have the same direction.) The most general form is

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$$

  where $\mathbf{a}$ is the point it passes through and $\mathbf{d}$ is the direction vector (i.e. the vector that points *along* the line).

- We can therefore show that the equation of the line through $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, because $\mathbf{b} - \mathbf{a}$ is the vector that travels *from* $\mathbf{a}$ *to* $\mathbf{b}$ <u>along the line</u>. For example find the line that passes through $(2, 3, 1)$ and $(3, 6, -1)$. This gives

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \qquad \text{or} \qquad \mathbf{r} = \begin{pmatrix} 3 \\ 6 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}.$$

- To find the angle between two vectors we use the scalar product result

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

  where $|\mathbf{u}|$ represents the magnitude of vector $\mathbf{u}$. From this we can see that two vectors are perpendicular if their scalar product is zero.

- The scalar product is most easily calculated as follows; $\begin{pmatrix} a_x \\ a_y \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \end{pmatrix} = a_x b_x + a_y b_y$. (It is just a number, *not* a vector!)

- The following table sums up the 3D equivalents of the 2D results we have already found:

2D	3D
$\mathbf{i}, \mathbf{j}$	$\mathbf{i}, \mathbf{j}, \mathbf{k}$
$\lvert\mathbf{a}\rvert = \sqrt{a_x + a_y}$	$\lvert\mathbf{a}\rvert = \sqrt{a_x + a_y + a_z}$
$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y$	$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$

  Most of the results from the 2D section (above) still hold true for 3D vectors.

- Obviously in 2D provided lines have different gradient then they *must* intercept somewhere. However in 3D it is possible for two lines to have different direction vectors (i.e. not be parallel) and still not cross: these lines are called *skew*. This example shows how to discover if lines in 3D intercept or are skew.

$$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix} 0 \\ -6 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

  Firstly we note the different direction vectors, so they cannot be parallel. Equate the x and y components[4] of both lines and solve for λ and μ:

$$\begin{aligned} (x): &\qquad 4 + \lambda = 2\mu, \\ (y): &\qquad -1 - \lambda = -6 + \mu. \end{aligned}$$

  These solve to $\lambda = 2$ and $\mu = 3$. Put these values back into the original lines and compare z-coordinates: if they are the same then they intercept, if different then skew.

$$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 8 \end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix} 0 \\ -6 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 8 \end{pmatrix}.$$

  Therefore the lines cross at $(6, -3, 8)$. (You should find that the x and y-coordinates are *always* the same, it is only the z-coordinate that might be different; a nice little check!)

- To find the angle between two lines then *dot their direction vectors*. Using the two lines in the above example we find:

$$\begin{aligned} \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} &= \sqrt{1^2 + 1^2 + 4^2}\sqrt{2^2 + 1^2 + 1^2} \cos\theta \\ 2 - 1 + 4 &= \sqrt{18}\sqrt{6}\cos\theta \\ \theta &= 61.2^\circ \text{ (to 3s.f.)}. \end{aligned}$$

  If you get an answer $90 < \theta \leqslant 180$ then give $180^\circ - \theta$ as your answer (Between any two lines there are two possible angles between them; think about it. The acute angle tends to be 'nicer').

- When working out angles in 3D you must be very careful that you are 'dotting' the right vectors! For example if $A = (1, 2, -2)$, $B = (3, 1, -4)$ and $C = (7, 5, 1)$ find the angle $A\hat{B}C$. Draw a sketch! We require the angle at B so we need to dot $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

  Now $\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ and $\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix}$. Therefore dotting we find:

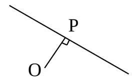
$$\begin{aligned} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix} &= \sqrt{9}\sqrt{57}\cos\theta \\ -8 + 4 + 10 &= 3\sqrt{57}\cos\theta \\ \frac{2}{\sqrt{57}} &= \cos\theta \qquad \Rightarrow \qquad \theta = 74.6^\circ \text{(to 3s.f.)}. \end{aligned}$$

[4] You can take any pair of components you like here (x & y, x & z, or y & z) but most students just take x and y.

- Some tough problems involve the use of

$$\text{“two vectors are at perpendicular”} \quad \Leftrightarrow \quad \text{“the dot product is zero”}.$$

For example find the point (P) on the line l: $\mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ closest to the origin.

Firstly draw a sketch of a line running some distance past an origin. 

At the point P the vector $\overrightarrow{OP}$ must be perpendicular to the line. The direction vector is the vector *along* the line l, so we need

$$(\overrightarrow{OP}).\begin{pmatrix}1\\-1\\2\end{pmatrix} = 0.$$

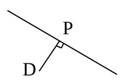
The point P is some point on the line, so $P = \begin{pmatrix}1+\lambda\\2-\lambda\\2\lambda\end{pmatrix}$ for some λ. So $\overrightarrow{OP} = \begin{pmatrix}1+\lambda\\2-\lambda\\2\lambda\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1+\lambda\\2-\lambda\\2\lambda\end{pmatrix}$. Therefore

$$\begin{pmatrix}1+\lambda\\2-\lambda\\2\lambda\end{pmatrix}.\begin{pmatrix}1\\-1\\2\end{pmatrix} = 0.$$

This gives $1+\lambda-2+\lambda+4\lambda = 0$ which solves to $\lambda = \frac{1}{6}$. Putting this λ back into l we find

$$P = (\tfrac{7}{6}, \tfrac{11}{6}, \tfrac{1}{3}).$$

- Another tough example done in two ways: Find the shortest distance from point $D = (2,-1,3)$ to the line $\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$.

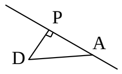
  - **Method I**: First method similar to above. Draw a sketch! 

Let the point on the line closest to D be P. So $P = \begin{pmatrix}1+2\lambda\\-\lambda\\1+\lambda\end{pmatrix}$. We require $\overrightarrow{DP}$ to be perpendicular to the line if it is the closest point. Thus

$$\overrightarrow{DP}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix} = 0 \quad \Rightarrow \quad \left(\begin{pmatrix}1+2\lambda\\-\lambda\\1+\lambda\end{pmatrix} - \begin{pmatrix}2\\-1\\3\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix} = 0.$$

This solves to $\lambda = \frac{5}{6}$. Therefore $P = (\frac{8}{3}, -\frac{5}{6}, \frac{11}{6})$. Therefore the distance is

$$\text{dist.} = |\overrightarrow{DP}| = |\mathbf{p}-\mathbf{d}| = \sqrt{(2-\tfrac{8}{3})^2 + (\tfrac{5}{6}-1)^2 + (3-\tfrac{11}{6})^2} = \frac{\sqrt{66}}{6}.$$

– **Method II**: Again, draw a sketch. 

Let P be the point closest to D. This time also include the point that we know the line passes through $A = (1, 0, 1)$. We have therefore created a right angled triangle APD with a right angle at P. Length AD is just the magnitude of $\mathbf{d} - \mathbf{a}$; $|\mathbf{d} - \mathbf{a}| = \sqrt{(2-1)^2 + (-1-0)^2 + (3-1)^2} = \sqrt{6}$. Angle $D\hat{A}P$ can be worked out by $\overrightarrow{AD} \cdot$ (direction vector). So

$$(\mathbf{d} - \mathbf{a}) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = |\mathbf{d} - \mathbf{a}| \left| \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right| \cos\theta$$

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \sqrt{6}\sqrt{6}\cos D\hat{A}P.$$

So $\cos D\hat{A}P = \frac{5}{6}$.

By right angled trigonometry $\sin D\hat{A}P = \frac{DP}{\sqrt{6}}$. To convert a sin into a cos we use $\sin^2\theta + \cos^2\theta = 1$ which gives $\sin D\hat{A}P = \frac{\sqrt{11}}{6}$. Therefore $DP = \sqrt{6} \times \frac{\sqrt{11}}{6} = \frac{\sqrt{66}}{6}$, just as before.