Core 4 Module Revision Sheet

The C4 exam is 1 hour 30 minutes long and is in two sections.

Section A (36 marks) 5 – 7 short questions worth at most 8 marks each.

Section B (36 marks) 2 questions worth about 18 marks each.

You are allowed a graphics calculator.

Before you go into the exam make sure you are fully aware of the contents of the formula booklet you receive. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted... always an area prone to error.

J M S

I CANNOT STRESS ENOUGH THAT WITH A2 MODULES SUCH AS C3 AND C4 SUCCESS IS ENTIRELY THROUGH PRACTICE. THE TOPICS AND QUESTIONS ARE HARDER AND LONGER AND ONLY DILIGENT PRACTICE WILL LET YOU GAIN TOP GRADES. THIS REVISION SHEET WILL THEREFORE BE OF ONLY LIMITED USE. *You have been warned!!!*

1. Algebra

- Review binomial expansion from C1 for $(x+y)^n$ for integer n. Notice that it is valid for *any* x and y and that the expansion has $n+1$ terms.
- The general binomial expansion is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

  and is valid for any n (fractional or negative) but $-1 < x < 1$ (i.e. $|x| < 1$). Notice also it must start with a 1 in the brackets. For example expand $(4-x)^{-1/2}$.

$$\begin{aligned}(4-x)^{-1/2} &= \left(4\left(1-\frac{x}{4}\right)\right)^{-1/2}\\ &= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}\\ &= \frac{1}{2}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{x}{4}\right)^2+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{x}{4}\right)^3+\cdots\right]\\ &= \frac{1}{2}\left[1+\frac{x}{8}+\frac{3x^2}{128}+\frac{15x^3}{3072}+\cdots\right].\end{aligned}$$

  It is only valid for $|x/4| < 1 \quad \Rightarrow \quad |x| < 4$.

- Partial fractions is effectively the reverse of combining together two algebraic fractions. For example

$$\frac{1}{x+1}+\frac{1}{x+2} \quad \begin{matrix}\longrightarrow & \text{Algebraic Fractions} & \longrightarrow\\ \longleftarrow & \text{Partial Fractions} & \longleftarrow\end{matrix} \quad \frac{2x+3}{(x+1)(x+2)}.$$

  To review algebraic fractions quickly; you must be able to manipulate algebraic fractions fluently. The rules for fractions are:

Adding/Subtracting	Multiplying	Dividing
$\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd}$,	$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$,	$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.

So for example simplify the following:

$$\frac{a}{a+1}+\frac{1}{a-1}=\frac{a(a-1)+1(a+1)}{(a+1)(a-1)}=\frac{a^2+1}{a^2-1}.$$

- The general principle for partial fractions is that we need to split an algebraic fraction into two (or more) fractions. So we create an *identity* that must be true for all x and then find certain missing constants. For example we expand

$$\frac{x-1}{(x-4)(x+1)} \text{ to } \frac{A}{x-4}+\frac{B}{x+1} \qquad \Rightarrow \qquad \frac{x-1}{(x-4)(x+1)}\equiv\frac{A}{x-4}+\frac{B}{x+1}.$$

We need to discover A and B. Multiplying through by $(x-4)(x+1)$ and cancelling we find

$$x-1\equiv A(x+1)+B(x-4).$$

We can either multiply out and equate coefficients[1] or choose values of x to work out the constants A and B. I prefer the latter. Here we would let $x=-1$ to discover $B=2/5$ and let $x=4$ to discover $A=3/5$. Therefore

$$\frac{x-1}{(x-4)(x+1)}\equiv\frac{3}{5(x-4)}+\frac{2}{5(x+1)}.$$

- The overall methods you need to know are:

$$\frac{px+q}{(ax+b)(cx+d)}\equiv\frac{A}{ax+b}+\frac{B}{cx+d},$$
$$\frac{px^2+qx+r}{(ax+b)(cx^2+d)}\equiv\frac{A}{ax+b}+\frac{Bx+C}{cx^2+d},$$
$$\text{and } \frac{px^2+qx+r}{(ax+b)(cx+d)^2}\equiv\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2}.$$

2. Trigonometry

- By definition

$$\sec x\equiv\frac{1}{\cos x}, \qquad \operatorname{cosec} x\equiv\frac{1}{\sin x}, \qquad \cot x\equiv\frac{1}{\tan x}.$$

I remember these by the third letter (underlined). You must be able to produce graphs of these in both radians and degrees (P184).

- By dividing $\sin^2 x+\cos^2 x\equiv 1$ by $\sin^2 x$ and $\cos^2 x$ we can derive $1+\cot^2 x\equiv\operatorname{cosec}^2 x$ and $\tan^2 x+1\equiv\sec^2 x$ respectively. These allow us to solve a whole family of equations that we couldn't solve before. With any equation with a combination of trigonometric functions (and if one of them is squared) we should be using these relationships to help us.

- It is important to note that is you must be careful when you see things like $2\tan x\sin x=\tan x$. It is *so* tempting to divide by $\tan x$ to yield $2\sin x=1$. But you must bring everything to one side and factorise; $\tan x(2\sin x-1)=0$. The full solutions can then found by solving $2\sin x-1=0$ <u>and</u> $\tan x=0$. [It is completely analogous to $x^2=x$. If we divide by x we find $x=1$, but we know this has missed the solution $x=0$. However when we factorise we find $x(x-1)=0$ and both solutions are found.]

[1] Here we would find $x-1\equiv Ax+A+Bx-4B\equiv(A+B)x+(A-4B)$. Therefore $A+B=1$ and $A-4B=-1$. These solve to $A=3/5$ and $B=2/5$.

3. Parametric Equations

- Curves are usually given in the form $y = f(x)$. However we have seen in C1 that this is not always the best form; for example a circle of centre (a, b) and radius r is usually given in the form $(x - a)^2 + (y - b)^2 = r^2$.

- A curve can also be expressed by a *parameter* (usually t or θ). In this system $y = f(t)$ and $x = f(t)$. Therefore instead of y being linked *directly* with x, they are linked *indirectly* through t. Usually t can vary over all the real numbers, but it can also be restricted (e.g. $t \geqslant 0$ or $0 \leqslant \theta \leqslant 2\pi$). Graphs of more exotic relationships can be obtained in parametrics (see P225). To graph the parametric you just vary the parameter and plot the resulting (x, y) points.

- You must be able to convert any reasonable parametric system to a normal xy relationship by eliminating the parameter. If it contains merely a polynomial relationship then it should be easy enough. For example eliminate t from $y = (t + 1)^2$, $x = t - 1$. From the x relationship $t = x + 1$, so $y = ((x + 1) + 1)^2 = x^2 + 4x + 4$.

- In harder examples you must use algebraic trickery; for example $y = \frac{t}{1-t}$, $x = \frac{t}{1+t}$. From the second we discover

$$x = \frac{t}{1+t} \qquad \Rightarrow \qquad x + xt = t \qquad \Rightarrow \qquad t = \frac{x}{1-x},$$

so by substitution

$$y = \frac{\frac{x}{1-x}}{1 - (\frac{x}{1-x})} = \frac{\frac{x}{1-x}}{\frac{1-x}{1-x} - (\frac{x}{1-x})} = \frac{x}{1-2x}.$$

- If the relationship involves sin's and cos's then you will need to be more cunning. Bear in mind that most of these will be circles or ellipses so we will probably be dealing with x^2's and y^2's. Always think about the ways we know to get rid of trig functions like $\sin^2\theta + \cos^2\theta = 1$ and the other trig identities from the previous section.

- For example eliminate t from $y = 1 + 3\sin\theta$, $x = 4 + 3\cos\theta$. We find

$$(y - 1)^2 = 9\sin^2\theta \qquad \text{and} \qquad (x - 4)^2 = 9\cos^2\theta.$$

Adding and using that $9\sin^2\theta + 9\cos^2\theta = 9$ we find $(x - 4)^2 + (y - 1)^2 = 9$. So it is a circle. This is a specific example of a general type you must learn:

$$\begin{aligned} x &= a + r\cos\theta \\ y &= b + r\sin\theta \end{aligned} \qquad \Longleftrightarrow \qquad (x - a)^2 + (x - b)^2 = r^2.$$

The similar relationship for the circle centre $(0, 0)$ is trivial from the above if we let $a = b = 0$.

- Calculus is possible with a parametric relationship. We know from the chain rule that $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$. A simple extension of this shows that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

For example find dy/dx for $x = e^{2t} + 1$, $y = 3e^{4t}$;

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12e^{4t}}{2e^{2t}} = \frac{6(e^{2t})^2}{e^{2t}} = 6e^{2t}.$$

- The questions asked on this can be rather algebraic! Instead of giving you a specific point on the curve they get you to consider a general point where the parameter is some value (say p). They then get you to consider the gradient of the tangent (or normal) at that point. You may then need to substitute this into $y - y_1 = m(x - x_1)$. There are two detailed examples on P238/9. Please try to do these by yourself; they are not easy.

- Obviously turning points are found by putting $dy/dx = 0$ (when aren't they?!). Once the value of the parameter is found, put it back into the original relationships to find the cartesian coordinates. To discover their nature you must go back to basics and consider points either side; don't try to do double differentiation[2].

4. Further Integration

- $\int_a^b \pi y^2\,dx$ is the volume of revolution of the curve y rotated about the x-axis between $x = a$ and $x = b$. All that is needed for you to do is calculate y^2 in terms of x from y. (For volumes of revolution around the the y-axis switch the x and the y and use $\int_p^q \pi x^2\,dy$ between $y = p$ and $y = q$.)

- You must be on the lookout for integrals of the form $\int \frac{1}{(x-1)(x-2)}\,dx$. Use partial fractions to split the terms and *then* integrate.

$$\int \frac{1}{(x-1)(x-2)}\,dx = \int \frac{1}{x-2} - \frac{1}{x-1}\,dx = \int \frac{1}{x-2}\,dx - \int \frac{1}{x-1}\,dx = \ln\left(\frac{x-2}{x-1}\right) + c.$$

  You must be very fluent with using partial fractions to simplify terms.

- The area under *any* curve can be *approximated* by the Trapezium Rule. The governing formula is given by (and contained in the formula booklet you will have in the exam)

$$\int_a^b y\,dx \approx \frac{1}{2}h\left[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\right],$$

  where h is the width of each trapezium, y_0 and y_n are the 'end' heights and $y_1 + y_2 + \cdots + y_{n-1}$ are the 'internal' heights.

5. Vectors

Two Dimensional Vectors

- The vector $\binom{3}{4}$ can be written $3\mathbf{i} + 4\mathbf{j}$ and represents a vector going 3 right and 4 up. By Pythagoras' Theorem it can be shown that the magnitude of this vector is $\sqrt{3^2 + 4^2} = 5$ and by trigonometry the direction is $\tan^{-1}\frac{4}{3}$. You must be fluent in converting between $\binom{x}{y}$ and (r, θ).

- Two vectors are equal if their magnitudes and directions are the same. Two vectors are parallel if one is a scalar multiple of the other. For example show that $\binom{2}{3}$ is parallel to $\binom{3}{4.5}$; so show that $1.5 \times \binom{2}{3} = \binom{3}{4.5}$.

- When multiplying a vector by a positive scalar it changes the length of the vector but not the direction. If the scalar is negative then it also reverses the direction of the vector. For example $3\binom{2}{-3} = \binom{6}{-9}$.

- When adding vectors, you just add the x components and add the y components. For example $\binom{2}{-3} + \binom{2}{5} = \binom{4}{2}$.

[2] It can be done, but it's fiddly!

- You must be good at the type of questions (similar to those from GCSE) on page 286. Know the geometric interpretation of $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$. Also know that in general if you have position vectors $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$ then $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$.

- A unit vector can be constructed by dividing a vector by its magnitude. For example the unit vector from $\binom{2}{3}$ is $\frac{1}{\sqrt{2^2+3^2}}\times\binom{2}{3}=\frac{1}{13}\times\binom{2}{3}$.

- A line can be written in vector form. If you know a line goes through a point (a,b) and has the gradient m then its vector form is $\mathbf{r}=\binom{a}{b}+\lambda\binom{1}{m}$ where λ is a scalar that takes different values on different points on the line. The vector $\binom{1}{m}$ can be re-written to make the components 'nicer'. For example $\binom{1}{2/3}\Rightarrow\binom{3}{2}$. (These vectors are not equal, but they have the same direction.) The most general form is

$$\mathbf{r}=\mathbf{a}+\lambda\mathbf{u}$$

  where $\mathbf{a}$ is the point it passes through and $\mathbf{u}$ is the direction.

- We can therefore show that the equation of the line through $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})=(1-\lambda)\mathbf{a}+\lambda\mathbf{b}$ (see page 294). These two ideas can be combined to convert to $y=mx+c$ form and vice-versa.

- To find the angle between two vectors we use the scalar product result

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$$

  where $|\mathbf{u}|$ represents the magnitude of vector $\mathbf{u}$. From this we can see that two vectors are perpendicular if their scalar product is zero.

- The scalar product is most easily calculated as follows; $\binom{a_x}{a_y}\cdot\binom{b_x}{b_y}=a_xb_x+a_yb_y$. (It is just a number, *not* a vector!)

Three Dimensional Vectors

- The following table sums up the 3D equivalents of the 2D results we have already found:

2D	3D
$\mathbf{i},\mathbf{j}$	$\mathbf{i},\mathbf{j},\mathbf{k}$
$\lvert\mathbf{a}\rvert=\sqrt{a_x^2+a_y^2}$	$\lvert\mathbf{a}\rvert=\sqrt{a_x^2+a_y^2+a_z^2}$
$\mathbf{a}\cdot\mathbf{b}=a_xb_x+a_yb_y$	$\mathbf{a}\cdot\mathbf{b}=a_xb_x+a_yb_y+a_zb_z$

  Most of the results from the 2D section (above) still hold true for 3D vectors.

6. Differential Equations

- If you are told that y is proportional to x then we write $y\propto x$. This implies that $y=kx$ for some *constant* k. k can then be determined by putting in one pair of values (x,y) into the equation.

- The notation dy/dx lets us believe the it is a normal fraction. Although this is not the case we can manipulate it like a fraction in a differential equation. You must move the variables to different sides of the equation and integrate (separation of variables). Only add the ever-present "$+c$" to one side. For example solve

$$\frac{dy}{dx}=y^2\cos x \qquad\Rightarrow\qquad \int\frac{1}{y^2}\,dy=\int\cos x\,dx \qquad\Rightarrow\qquad y=-\frac{1}{\sin x+c}.$$