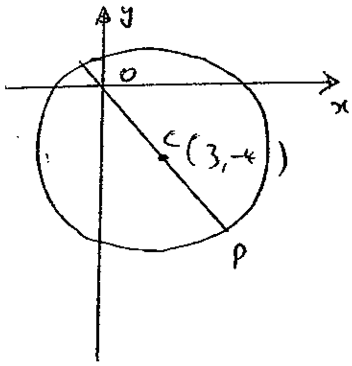
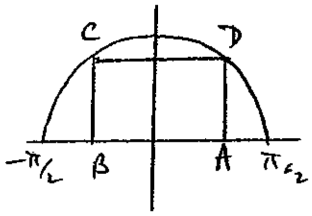
June 2005
9801 AEA
Mark Scheme

Question Number	Scheme	Marks
1	$(x-3)^2+(y+4)^2 = 24+9+16 = 49$	M1
	Curve is circle, centre $(3,-4)$, radius 7	A1
	 $OC = \sqrt{3^2+4^2} = 5$	M1
	Greatest length $OP = 5+r$ (or least)	M1
	$= \underline{12}$	A1
	Least length $= r-5 = \underline{\underline{2}}$	A1
		(6)

Question Number	Scheme		Marks
2	$2\sin\theta\cos\theta + \cos 2\theta + 1 = \sqrt{6}\cos\theta$	(use of $\sin 2\theta =$)	M1
	$2\sin\theta\cos\theta + 2\cos^2\theta = \sqrt{6}\cos\theta$	($2\cos^2\theta$ needed)	M1
	$\cos\theta\,(2\sin\theta + 2\cos\theta - \sqrt{6}) = 0$		
	$\cos\theta = 0 \Rightarrow \theta = \underline{\pi/2,\ 3\pi/2}$	(Factor of $\cos\theta$) (both)	M1 A1
	or $\sin\theta + \cos\theta = \frac{\sqrt{6}}{2}$		
	$\sqrt{2}\sin(\theta + \pi/4) = \frac{\sqrt{6}}{2}$	(use of $\sin(\theta+\pi/4)$ or $\cos(\theta-\pi/4)$)	M1
	$\sin(\theta + \pi/4) = \frac{\sqrt{3}}{2}$		A1
	$\theta + \frac{\pi}{4} = \frac{\pi}{3}$ or $\frac{2\pi}{3}$	(2 values)	M1
	$\underline{\theta = \frac{\pi}{12},\ \frac{5}{12}\pi}$	(both)	A1
			(8)

3	$\frac{d}{dx}(u\sqrt{x}) = \sqrt{x}\frac{du}{dx} + \frac{1}{2\sqrt{x}}.u$ (product rule)	M1
	$\therefore$ Eqn $\Rightarrow$ $\sqrt{x}\frac{du}{dx} + \frac{1}{2\sqrt{x}}.u = \frac{1}{2\sqrt{x}}\frac{du}{dx}$ (D.E.) (all $\frac{d}{dx}$'s completed)	A1
	$\therefore \frac{du}{dx}\left(\frac{1}{2\sqrt{x}} - \sqrt{x}\right) = \frac{1}{2\sqrt{x}}u$	
	$\therefore \frac{du}{dx}(1-2x) = u$ (simplification)	M1, A1
	$\therefore \int\frac{du}{u} = \int\frac{dx}{1-2x}$ (sepn of variables)	M1
	$\therefore \ln u = -\frac{1}{2}\ln(1-2x)\ \left[+\frac{1}{2}\ln K\right]$ (int.n) (or $\ln(2x-1)$)	M1 (2 int terms) A1 (cao)
	~~Since $0 < x < \frac{1}{2}$, $\ln u = -\frac{1}{2}\ln(1-2x) + \frac{1}{2}\ln K$~~	
	~~$u^2 = \frac{K}{1-2x}$~~	
	$u = 4,\ x = 3/8 \Rightarrow 16 = \frac{K}{1-3/4} \Rightarrow K = 4$ (use of condition)	M1
	$\therefore \underline{u = 2(1-2x)^{-1/2}}$	A1
		(9)

4 (a)



By symmetry, B is $(-p, 0)$ — M1

Area $= \underline{2p\cos p}$ — A1 (2)

(b) $\dfrac{dA}{dp} = 2\cos p - 2p\sin p$ — B1

$\dfrac{dA}{dp} = 0 \Rightarrow 1 = p\tan p$, so when $p = \alpha$, $\underline{\alpha\tan\alpha = 1}$ o.e. — B1

[This mark can be earned in (c)]

Let $f(p) = p\tan p - 1$ (o.e.)

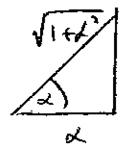
$f(\pi/4) = \pi/4 - 1 < 0$ — (f at one end) M1, ($f(\pi/4)<0$) A1

$f(1) = \tan 1 - 1 = \tan 1 - \tan \pi/4$

$(1 > \pi/4)$ $\therefore \tan 1 - \tan\pi/4 > 0$ — (Reason $f(1)>0$) A1

change of sign $\therefore$ $\underline{\pi/4 < \alpha < 1}$ — A1 (6)

(c) Max. area $= 2\alpha\cos\alpha$, with $\tan\alpha = \dfrac{1}{\alpha} \Rightarrow$



M1 (Δ or trig)

$= \underline{\dfrac{2\alpha^2}{\sqrt{1+\alpha^2}}}$ $(= S)$ — A1 cso (2)

(d) $\dfrac{dS}{d\alpha} = \dfrac{4\alpha\sqrt{1+\alpha^2} - 2\alpha^3/\sqrt{1+\alpha^2}}{1+\alpha^2} = \dfrac{4\alpha + 2\alpha^3}{(1+\alpha^2)^{3/2}} > 0$ — M1

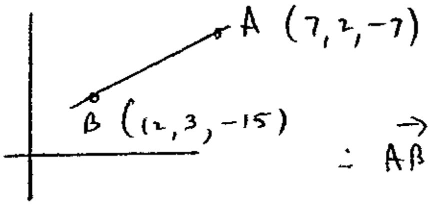
$\therefore$ S is an increasing function as α varies

S lies between $\dfrac{2\left(\frac{\pi}{4}\right)^2}{\sqrt{1+\left(\frac{\pi}{4}\right)^2}} = \dfrac{\pi^2}{2\sqrt{16+\pi^2}}$ (subst. $\alpha = 1$ and $\alpha = \pi/4$) — M1

and $\dfrac{2(1)^2}{\sqrt{1+1}} = \sqrt{2}$

i.e. $\underline{\dfrac{\pi^2}{2\sqrt{16+\pi^2}} < S < \sqrt{2}}$ — A1 (cso) (3)

5 (a)



$\vec{AB} = (5, 1, -8)$

$\therefore \vec{AB}$ is $\underline{r} = \begin{pmatrix} 7 \\ 2 \\ -7 \end{pmatrix} \pm \lambda \begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix}$ — M1, A1 (2)

(b) $\mu = 4 \Rightarrow \underline{r} = (0, 0, 0)$ $\therefore$ L_2 passes through O — B1 (1)

(c) If intersect,

$$\left.\begin{aligned} 7 + 5\lambda &= -4 + \mu \\ 2 + \lambda &= 0 \\ -7 - 8\lambda &= 12 - 3\mu \end{aligned}\right\}$$

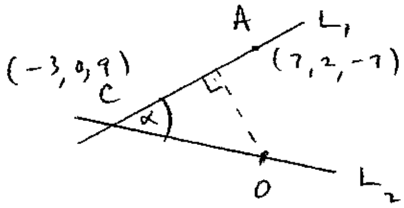
(Any 2 eqn's) — B1

Solving $\Rightarrow \lambda = -2$, $\mu = 1$ (solving 2 eqn's) — M1

check in third equation ($7 - 10 = -4 + 1$ or $-7 + 16 = 12 - 3$)

$\therefore \vec{OC} = \begin{pmatrix} -3 \\ 0 \\ 9 \end{pmatrix}$ (check 3rd eqn + answer) — A1 (3)

(d)



$\vec{AC} = (-10, -2, 16)$ (or any vector along L_1) ($\vec{AC}$, allow ±) — M1

$\vec{AC} \cdot \vec{OC} = 30 + 0 + 144 = 174$

$= \sqrt{360} \cdot 3\sqrt{10} \cos\alpha$

$\therefore \cos\alpha = \dfrac{174}{6 \times 10 \times 3} = \dfrac{29}{30}$ (dep) — M1, A1 (3)

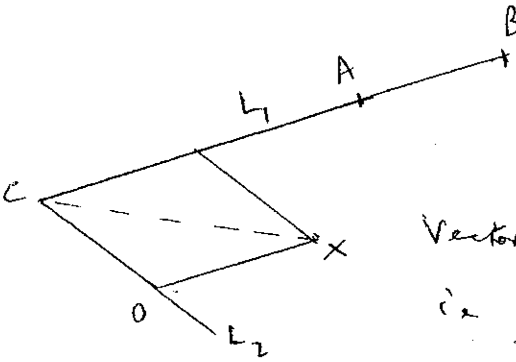
(e) Shortest distance $= |OC| \sin\alpha = 3\sqrt{10} \cdot \sqrt{1 - \left(\frac{29}{30}\right)^2}$ (complete method for M2)

$= 3\sqrt{10} \cdot \dfrac{\sqrt{59}}{3\sqrt{10}\sqrt{10}} = \sqrt{\dfrac{59}{10}}$ o.e. — M2, A1 (3)

(f) $|\vec{CO}| = 3\sqrt{10}$; $|\vec{AB}| = \sqrt{25 + 1 + 64} = 3\sqrt{10}$ — M1 (both lengths)

$\therefore |\vec{CO}| = |\vec{AB}|$ — A1 (2)

(g)



$\vec{CX} = \vec{CO} + \vec{OX}$ $[= \vec{CO} + \vec{AB}]$ — M1

$= \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -17 \end{pmatrix}$ (dep) — M1, A1

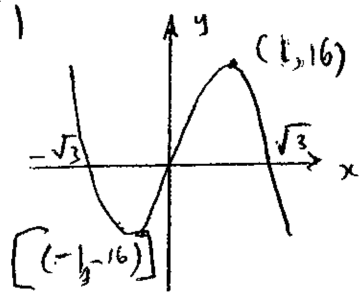
Vector eqn of bisector $\underline{r} = \vec{OC} + t\,\vec{CX}$ — M1 (dep)

i.e. $\underline{r} = \begin{pmatrix} -3 \\ 0 \\ 9 \end{pmatrix} \pm t \begin{pmatrix} 8 \\ 1 \\ -17 \end{pmatrix}$ — A1 (5)

6 (a) $f = 0,\ x^2 = 12 \quad \therefore \quad R$ and P are $\underline{(\pm\sqrt{12}, 0)}$ — B1, B1

$f' = 12 - 3x^2 \quad f' = 0,\ x^2 = 4\ ;$ so $\underline{Q = (2, 16)}$ — M1, A1 (4)

(b) (i)

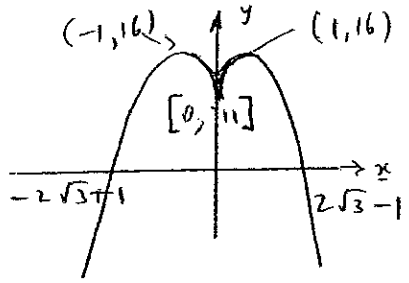


shape, symmetry about O — B1

Any one of P', R', Q' — M1

all correct — A1 ✓

(ii)



shape, symmetry about axis — B1

$(2\sqrt{3}-1, 0)$ and $(1, 16)$ & meets y-axis above origin — B1

$(-2\sqrt{3}+1, 0)$ and $(-1, 16)$ — B1 (6)

(c) Min. at $(-2, -16) \Rightarrow (\alpha, 0)$

$\therefore$ curve has "moved up" by 16 $\quad \therefore \quad \underline{w = 16}$ — M1, A1

$x = 0 \Rightarrow \quad f(-v) + w = 2 \times 16$ ($\because$ S, U have same y-coordinates) — M1

$\therefore f(-v) = 16$

$-v(12 - v^2) = 16$

$v^3 - 12v - 16 = 0$ (correct cubic in v) — A1

$(v + 2)^2(v - 4) = 0$ (finding a root) — M1

$\therefore v = -2$ or $v = 4$

Min. has moved from −2 to +ve value, so $v > 0$

$\therefore \underline{v = 4}$ — A1

Horizontal movement of 4 so $\underline{T \text{ is } (2, 0)}$ — B1

$S = (0, 32)$; $T = (2, 0)$; $U = (6, 32)$ so by symmetry other point of intersection is $\underline{x = 8}$; — M1, A1 (9)

6 (c)	Alternative solution. (algebraic)	
	Let $g(x) = f(x-v) + w$	
	$= (x-v)(12-(x-v)^2) + w$	
	$= 12(x-v) - (x-v)^3 + w$	
	$g'(x) = 12 - 3(x-v)^2 = 0$ when $(x-v)^2 = 4$	
	$x - v = \pm 2$	
	Smaller of these values is the min. $\therefore$ $x = v-2$ at min.	
	$g(v-2) = -24 + 8 + w = 0$ $\therefore$ $\underline{w = 16}$ (complete method for M1)	M1, A1
	$g(0) = -12v + v^3 + 16$	
	$g(v+2) = 24 - 8 + 16 = 32$ (max. point)	
	$g(0) = g(v+2)$ $\therefore$ $v^3 - 12v - 16 = 0$ ($g(0)=g(v+2)$)	M1
	$(v+2)^2(v-4) = 0$ (correct cubic)	A1
	$\therefore$ $v = -2$ or $v = 4$ (a root)	M1
	$v = -2$ means min is at $(-4, 0)$ — moved to the left, so not valid $\therefore$ $\underline{v = 4}$	A1
	$\therefore$ $\underline{T \text{ is } (2, 0)}$ ie $\underline{x = 2}$	B1
	$g(x) = 12(x-4) - (x-4)^3 + 16$	
	$\Rightarrow$ $x^3 - 12x^2 + 36x - 32 = 0$	
	$(x-2)^2(x-8) = 0$	
	$\therefore$ $\underline{\text{other root is } x = 8}$	M1, A1
		(9)

7 (a)	$x = \sec\theta$; $dx = \sec\theta\tan\theta\, d\theta$	M1
	$I = \int (\sec^2\theta - 1)^{1/2} \sec\theta\tan\theta\, d\theta$ $(\sec^2\theta - 1)$	M1
	$= \int \sec\theta\tan^2\theta\, d\theta$	A1 (3)
(b)	$J = \int \tan\theta(\sec\theta\tan\theta\, d\theta)$ (correct u, v)	M1
	$= \sec\theta\tan\theta, - \int \sec\theta . \sec^2\theta\, d\theta$	A1, A1
	$= \sec\theta\tan\theta - \int \sec\theta(1+\tan^2\theta)\, d\theta$ (split $\sec^3$)	M1, A1
	$= \sec\theta\tan\theta - \int \sec\theta\, d\theta - J$	
	$\therefore 2J = \sec\theta\tan\theta - \ln\lvert\sec\theta + \tan\theta\rvert$ $(+c)$	M1 (collect J's)
	$\therefore J = \frac{1}{2}\left[\sec\theta\tan\theta - \ln\lvert\sec\theta + \tan\theta\rvert\right]$ [+ const]	A1 cso (7)
(c)	$K = \int_0^{\pi/4} \sin x\sqrt{2\cos^2 x - 1}\, dx$	M1
	* $v = \sqrt{2}\cos x$; $dv = -\sqrt{2}\sin x\, dx$ (v, dv both needed)	M1
	$\therefore K = -\frac{1}{\sqrt{2}}\int_{\sqrt{2}}^{1} \sqrt{v^2 - 1}\, dv$ (integrand)	A1
	$v = \sec\theta$; $dv = \sec\theta\tan\theta\, d\theta$ $(\sec\theta)$	M1
	$K = -\frac{1}{\sqrt{2}}\int_{\pi/4}^{0} \sec\theta\tan^2\theta\, d\theta$ (limits) (integrand)	M1 A1 $\surd$
	$= \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\left[\sec\theta\tan\theta - \ln\lvert\sec\theta + \tan\theta\rvert\right]_0^{\pi/4}$	B1 $\surd$ their (b) or printed (b)
	$= \frac{1}{2\sqrt{2}}\left(\sqrt{2} - \ln(\sqrt{2}+1)\right)$ (use of limits)	M1
		A1 cao
	* Alternatively $v = \cos x$, followed by $\sqrt{2}v = \sec\theta$	(9)